ON σ-PROXIMITY, δ-LIQUIDITY, AND STABILITY METRICS AS INDICATORS FOR DOWNTURNS

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1. Introduction

Portfolio building is an essential part of building financial security for ones future. However, putting your assets in the market leaves you vulnerable to great loss when the market goes down. So the question that arises is are there indications prior to bad events in the market that tell us to be careful. We studied this questions in the context of the following assets from 2006-2021:

- VFINX - A fund that tracks the S&P500
- VBMFX - A fund that tracks the Bloomberg Barclays U.S. Aggregate Float Adjusted Index. It is mostly bonds.
- VGSLX - A fund that keeps track of the Vanguard Real Estate Index
- VSMAX - This fund invests in companies in the CRSP US Small Cap Index
- VEIEX - This fund invests in the FTSE Emerging Markets All Cap China A Inclusion Index.
- VGENX - This fund invests in common stock of companies in the energy industry.
- AMZM - Amazon is a company that does retail of consumer products. They also manufacture various electronic products as well as have a membership service that offers free shipping and access to streaming TV shows and movies.
- COST - Costco is a wholesale corporation that operates membership warehouses. They sell a variety of goods including groceries, electronics, and home goods.
- WMT - Walmart operates retail and wholesale stores around the world. They have a variety of stores such as supermarkets, supermarkets, and discount stores. They operate in 26 countries with a total of 11,400 stores worldwide.

We studied the following metrics as potential indicators of downturns in the market:

- σ-Proximity
- δ-Liquidity
- Stability

From 2006-2021 there were 6 events that caused significant downturns in the markets. In 2007-2008 there was the housing market crash. The European Debt Crisis caused crashes in both 2010 and 2012. The 2015 Chinese market crash. The trade war with China in 2018. And finally the market crash caused by the onset of COVID-19 in 2020. So when trying to determine whether any of our metrics could be leading indicators for market downturns we looked at how they behaved prior to these crashes and compared that to how the metric generally behaved. If the metric consistently did a 'dance' prior to the crashes and was less active when the market was more consistent that would be a good indicator. However, if the 'dance' of the metric is not unique to the crashes, or it did not do the 'dance' prior to most of the crashed then it may not be a good leading indicator.

2. Market Overview from 2006 to 2021

When analyzing ν for each year for each of the three groups we noticed that it was much smaller for Group A than AB and ABC across all years (Appendix A). This makes sense because ν is the slope of the asymptotes of the Markowitz frontier. When there are more assets in the portfolio the frontier is wider. When looking at stability metrics, we found that they are they are almost always higher for the larger groups (Appendix C). This makes sense as the assets in group A are relatively stable but the assets in group B and C are subject to a lot more instability.

When studying the Sharpe Metrics with respect to the assets in Group A we found that the metric was lowest for VFINX (Appendix D). This is explained by the fact that the Sharpe metric is dependent on the ratio of slope of risk free rate to the slope of the asymptote of the frontier. When studying long tangent portfolios we noticed that in years with significant downturns in the market often the allocation was to just one or two assets (Appendix G). We also noticed that downside potential was lower than in non tangent portfolios, as expected.

When looking at the sigma-mu planes, we noticed that the limited leverage frontiers have varying shape from year to year. This tells us that putting a limit on leverage will have a different affect on your portfolio each year.

The individual assets are typically clustered together close to the vertex of the frontier. In years where the market does poorly some assets may actually be pretty far along the efficient long frontier (Ex Amazon in 2015 and 2020, Appendix H).

In 2008, the individual assets are all very close to the inefficient frontier. Additionally, the long efficient frontier is very far away from the efficient frontier. This is because limiting the portfolio to no short positions in 2008 significantly hurt it. Conversely, when the market does really well, (Ex 2019, Appendix H), the limited leverage frontier is very close to the efficient frontier. However, the efficient frontier is very short indicating that if you want more than a certain return, you must take some short positions.

The computation for the two-rate limited leverage model were reaching too small values that the machine arithmetic has become imprecise. Increasing precision, with the quadprog command on our machines, leads to unfeasible computation times. We find that this is due to the fact that many values of the lower end of the μ-range for the frontier corresponds to extremely low σ values as the risk-free rates have 0 risk associated with them.

3. Metrics as Indicators

We use VFINX as our proxy for the performance of the market. We analyze plots of metrics over time in comparison to the monthly returns of VFINX. Thus, stock market crashes are indicated on the VFINX monthly returns plot as particularly low minima. From the VFINX plot, we then analyze both real world events and movement in the metrics over time plot to determine the efficacy of a metric as an indicator for crashes.

The metrics studied, σ-proximity, δ-liquidity, and the stability metrics, are calculated in a yearly rolling basis per month, starting at December 2006. In other words, month 1 of our analysis uses data from January 1, 2006 until December 31, 2006 to calculate the values for the metrics. Further, month 2 uses data from February 1, 2006 until January 31, 2007 to calculate the values for the metrics, and so on.

3.1. **Stock market crashes.** Before we dive into comparing the metrics to the performance of the stock market, we ought to first analyze the events that lead to the crashes. We expect, if any of the metrics are good indicators, to already begin their indication in response to the events leading up to the crashes. Thus, for each crash, a **critical period** will be defined where an analysis of the movement in the metrics is to be done so as to find an indication of the upcoming crash. These critical periods will be defined by analyzing the events leading up to the crash a priori.

The first crash we analyze is the Housing Market Crash of 2007-2008. The stock market saw its worst month in October of 2008 (month 23) following the collapse of the Lehman Brothers the month prior, but the events leading up to this crash occurred many months before. We see that by June 2007 (month 7) over 20 subprime lenders have filed for bankruptcy and Bear Stearns was beginning to show signs of financial fragility. And so, the critical period for this crash starts at month 7 and ends at month 23, when the crash finally occurred.

We perform a similar analysis of events for the five other crashes to define the critical periods which precede them. We see a crash in the stock market on May 2010 (month 42) which we believe to be in response to the European Sovereign Debt Crisis. This Debt Crisis began to shown signs in October 2009 (month 35) when Greece's deficit was revealed to be much more than expected. Hence, we define the critical period for this crash to be from month 35 to month 42.

We see another crash on May 2012 (month 66) which seems to also be in response to the European Debt Crisis. The signs for this crash seems to begin to show in August 2011 (month 57) when the European stock markets fall. And so, we define the critical period for this crash to be from month 57 to month 66.

Further, we find another crash on August 2015 (month 105) in response to a crash in the Chinese stock market. This crash follows a bubble growing in the Shanghai exchange beginning on June 2014 (month 91). The critical period for this crash is then defined from month 91 to month 105.

Finally, the two final crashes in response to the 2018 US-China trade war and the 2020 COVID pandemic were not particularly caused by movements in the financial system, but rather were caused by the whims of a politician and a force of nature. However, the financial system did move as a reaction to these events, and we can see indications prior to the crash itself. Thus, we will define the critical period for the 2018 US-China trade war to begin at January 2018 (month 134), when the US President began to impose tariff on China, until December 2018 (month 145) when the stock market crash happened. We will not define a critical period for the 2020 COVID pandemic crash, but instead analyze what occurred immediately prior and see if we can draw the same conclusions as we have for previous crashes.

The information here were synthesized from Investopedia's page on the 07-08 Financial Crisis, Wikipedia's entries on the 15-16 Chinese stock market turbuluence and 2018 Trade War, and Wikipedia's timeline on the EU debt crisis.

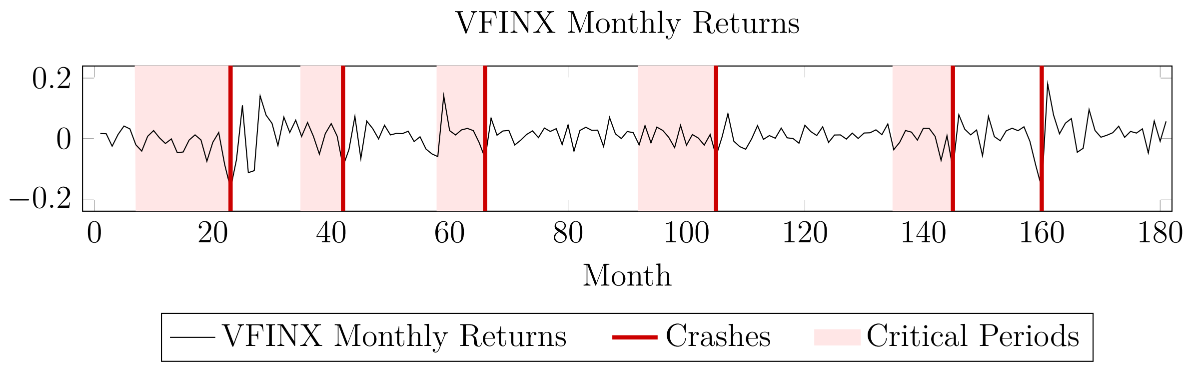


FIGURE 1. VFINX Monthly returns calculated from December 2006 (month 1) until December 2021 (month 181). The Red lines indicate the crashes discussed in this section, and the shaded red regions are critical periods preceding the crashes.

3.2. **δ-liquidity.** The δ-liquidity metric, ω^δ, is a metric that measures the downside potential (the worst daily returns) of a solvent portfolio. In particular, the metric compares the downside potential of portfolio over a fixed time period with its mean returns. It is close to 0 when the worst daily returns is close to the mean daily returns, and it is 1 when in the worst day of performance, the portfolio reached or went below a value of 0. Formally,

Definition 3.1 (δ-liquidity)
Given a portfolio $\mathbf{f}$ and a time period indexed daily by $d = 1, \ldots, D$. Let $\mu(\mathbf{f})$ be the mean daily returns of the portfolio over $d = 1, \ldots, D$. Define the downside potential, $\delta(\mathbf{f})$ to be:

$$\delta(\mathbf{f}) = \max\left\{-\mathbf{r}(d)^\intercal \mathbf{f} : d = 1, \ldots, D\right\}$$

where $\mathbf{r}(d)$ is the returns vector for day d. Finally, define the δ-liquidity metric to be:

$$\omega^\delta(\mathbf{f}) = \min\left\{1, \frac{\delta(\mathbf{f}) + \mu(\mathbf{f})}{1 + \mu(\mathbf{f})}\right\}$$

Before looking at the plots, we realize that due to the nature of δ, it is unlikely that this metric is a good indicator. We figured, more likely, that ω^δ will lag, and we will see jumps precisely when crashes happen. Further, the portfolio analyzed is where only long VFINX assets are held. We realize, across all our metrics, that the movement of the returns of VFINX will be most indicated by metrics measuring the performance of just VFINX.

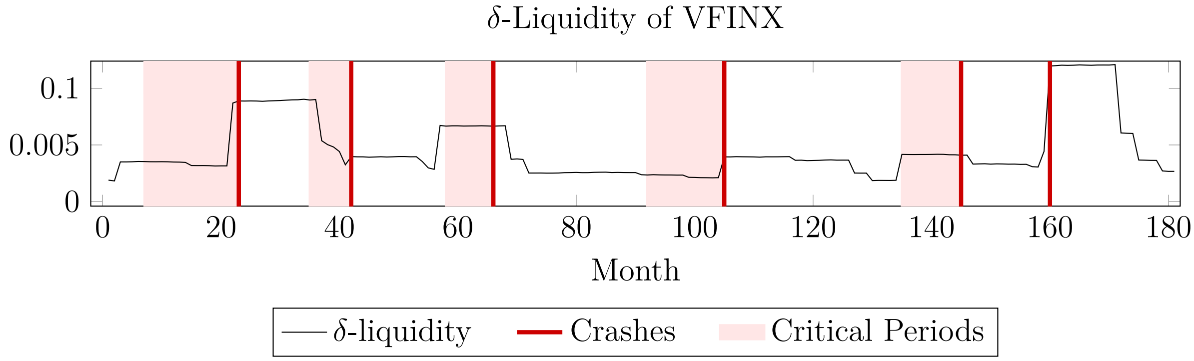


FIGURE 2. δ-liquidity for VFINX calculated from December 2006 (month 1) until December 2021 (month 181) using yearly rolling data. The red lines and shaded areas indicate the same idea as in Figure 1.

From Figure 2, we see no indication of crashes. Sometimes in the critical periods approaching a crash, we see a sharp increase in the metric, and at other times the sharp increase coincides with the crash. Sometimes, we see a decrease, and other times, we see it stay constant. This inconsistent behavior prior to crashes reveals that this metric is not a good indicator of stock market crashes.

3.3. **Stability Metrics.** The two stability metrics are both functions of what is known as the stability ratio: the ratio of the slope of the asymptote of the Markowitz frontier and the slope of the tangent formed by a risk-free rate. A measure of the stability of the market is how much returns does one need to expect so that the most efficient allocation stops/begins to require the use of a risk-free asset. This stability ratio is close to 0 when this expected returns is close to the minimum volatility return, and it approaches 1 as this expected returns grows.

Definition 3.2 (Stability Metrics)
Given a risk-free rate and a Markowitz frontier, define two metrics: the asymptote intersection metric, $\omega_{\text{rf}}^{\text{as}}$, and the tangent point metric, $\omega_{\text{rf}}^{\text{tg}}$ as

$$\omega_{\text{rf}}^{\text{as}} = \frac{\nu_{\text{mv}}}{\nu_{\text{rf}}}, \omega_{\text{rf}}^{\text{tg}} = \frac{\nu_{\text{mv}}^2}{\nu_{\text{rf}}^2}$$

where ν_{mv} is the slope of the Markowitz frontier's asymptote, and ν_{rf} is the slope of the tangent line connecting the Markowitz frontier and risk-free asset.

Before analyzing the plots, we figured that this could be a good metric as it measures the stability of the economy as a whole. However, we were also suspicious that it cannot take into special account the performance of VFINX as it is not a function of a portfolio. After all, this metric would remain unchanged if the performance of VFINX and the performance of VBMFX were to switch.

Further, we note that since $\omega_{\text{rf}}^{\text{tg}} = (\omega_{\text{rf}}^{\text{as}})^2$, they move in the same direction. Thus, we decide to only analyze one of them. Also, because of the close relation of the two risk-free rates – the credit line rate is annually just 0.03 more than the safe investment rate – we observed that the plots of the metrics with respect to either risk-free rate moved in a similar fashion.

We found that the metrics move generally in the same direction when using Group A's Markowitz frontier, Group AB's, or Group ABC's. However, we found that these movements were more pronounced for groups with fewer assets. Thus, we analyze the asymptote intersection metric with respect to Group A's Markowitz frontier so that it is easier to find patterns.

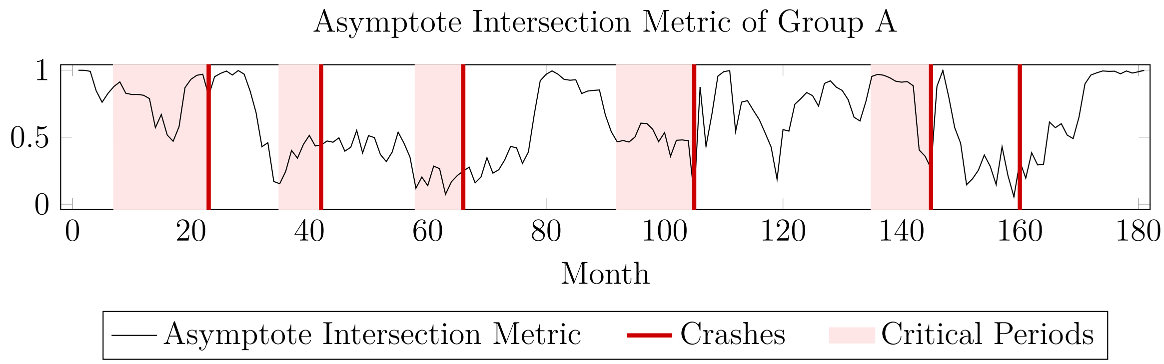


FIGURE 3. Asymptote Intersection Metric for Group A calculated from December 2006 (month 1) until December 2021 (month 181) using yearly rolling data. The red lines and shaded areas indicate the same idea as in Figure 1.

We see that some crashes are preceded with high values, others are preceded by low values. Further, we also do not see consistent movements in the plot. The month 23 crash was preceded immediately by a sharp increase, whereas the crash at month 145 was preceded immediately by a sharp decrease. And thus, we concluded that there seems to be no consistent pattern that indicates a crash for this metric.

3.4. **σ-proximity.** The σ-proximity metric, ω^σ, measures the amount of risk that can be reduced by reallocating a portfolio, the "diversifiable risk," while keeping the same reward. In particular, it is the ratio between the diversifiable risk of the portfolio over the total risk of the portfolio.

Definition 3.3 (σ-proximity)
Given a portfolio $\mathbf{f}$. Let $\sigma(\mathbf{f})$ be the volatility/risk associated with $\mathbf{f}$ calculated using its daily returns over a set time period. Let $\sigma_{\text{mv}}(\mathbf{f})$ be the volatility of the Markowitz portfolio with the same mean returns as $\mathbf{f}$, in other words, it is the minimum volatility one can achieve by reallocating $\mathbf{f}$ while keeping the same mean returns. Then, let the diversifiable risk of $\mathbf{f}$, $\sigma_{\text{dv}}(\mathbf{f})$, be the positive number such that,

$$(\sigma(\mathbf{f}))^2 = (\sigma_{\text{mv}}(\mathbf{f}))^2 + (\sigma_{\text{dv}}(\mathbf{f}))^2$$

Finally, we define the σ-proximity of $\mathbf{f}$, $\omega^\sigma(\mathbf{f})$, to be:

$$\omega^\sigma(\mathbf{f}) = \frac{\sigma_{\text{dv}}(\mathbf{f})}{\sigma(\mathbf{f})}$$

Initially, we figured that σ-proximity would not be the best candidate for indication. We figured that the volatility will only increase when large negative returns begin to enter the data. And so, we figured that if it were to be an indicator at all, it would be a late leading one or even a lagging one.

The portfolio whose σ-proximity we analyzed contains only long holdings for VFINX. We figured that, the use of other portfolios will muddy the movement of the indicator as their performance is already taken into account by the backing Markowitz frontier. Further, the σ-proximity of VFINX will in essence be a comparison of the volatility of VFINX compared to the rest of the economy. Also, we were wary of attempting to incorporate more variables when looking for an indication, especially since the Principality Component Analysis from our previous work suggests a dimension of 1.

As observed in the stability metric, the metric moved in similar directions when we varied the backing Markowitz frontier among Group A, Group AB, and Group ABC. The difference is that the movement of the metric is most pronounced when calculated using the Group A Markowitz frontier. And thus, we analyze the metric calculated using Group A's Markowitz frontier.

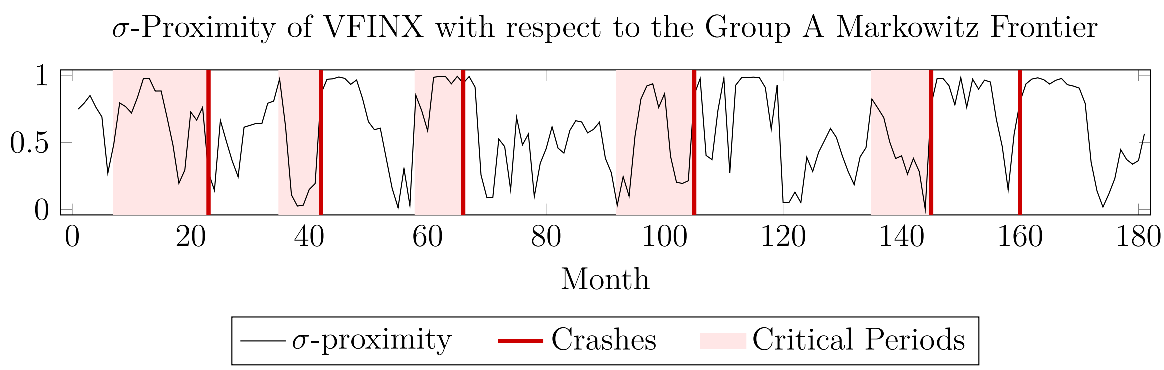


Figure 4. σ-Proximity of VFINX with respect to the Group A Markowitz Frontier calculated from December 2006 (month 1) until December 2021 (month 181) using yearly rolling data. The red lines and shaded areas indicate the same idea as in Figure 1.

It seems that there is a regular pattern in this metric. We see that during each critical period prior to a crash, the σ-proximity reaches close to 1. Then, we see a sudden drop from this high point preceding the crash. Some drops are not so steep, like the one seen before the month 66 crash, but there is still a marked drop nonetheless. Further, we see this pattern as well leading into the crash at month 160, even though as discussed, the crash was not financial in nature.

While the pattern seemed like a good indicator for a crash, what was bothersome is the sudden drop we witnessed prior to the crash. A lower value for σ-proximity hints at a more stable market. The key here is that, it is less about VFINX becoming less volatile, but rather the economy becoming more volatile, which means that the diversifiable volatility of VFINX decreases. We see this precede nearly every crash. As an example, here are Markowitz frontiers and VFINX during the critical period preceding the month 23 critical period.

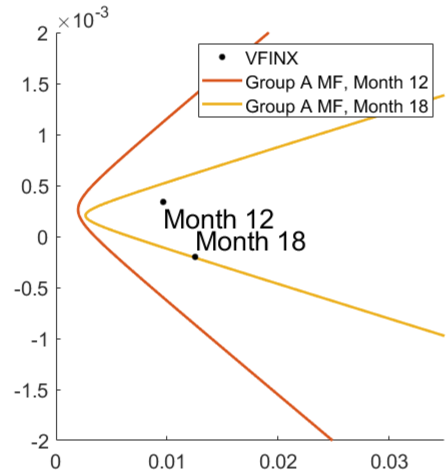


FIGURE 5. Group A Markowitz frontiers and VFINX on the $\sigma\mu$-plane for Month 12 and Month 18.

There is an exception to this explanation however. The months preceding the crash of month 42 saw the Markowitz frontiers widen, rather than become narrower. We attribute this to the fact that the rolling yearly data used to calculate these frontiers began to exclude data the extremely volatile and unprofitable markets from the 07-08 financial crisis.

Appendix A. Markowitz Frontier Parameters

	2006	2007	2008	2009	2010	2011	2012	2013
σ	0.002	0.002	0.003	0.003	0.002	0.002	0.001	0.002
μ	0	0	0	0	0	0	0	0
ν	0.112	0.088	0.083	0.047	0.056	0.041	0.057	0.2

	2014	2015	2016	2017	2018	2019	2020	2021
σ	0.001	0.002	0.002	0.001	0.002	0.002	0.003	0.002
μ	0	0	0	0	0	0	0	0
ν	0.122	0.013	0.041	0.134	0.019	0.102	0.068	0.153

Table 1. Group A Parameters

	2006	2007	2008	2009	2010	2011	2012	2013
σ	0.002	0.002	0.003	0.003	0.002	0.002	0.001	0.002
μ	0	0	0	0	0	0	0	0
ν	0.146	0.18	0.123	0.161	0.098	0.139	0.126	0.275

	2014	2015	2016	2017	2018	2019	2020	2021
σ	0.001	0.002	0.002	0.001	0.002	0.002	0.003	0.002
μ	0	0	0	0	0	0	0	0
ν	0.195	0.113	0.081	0.209	0.104	0.128	0.183	0.194

Table 2. Group AB Parameters

	2006	2007	2008	2009	2010	2011	2012	2013
σ	0.002	0.002	0.003	0.003	0.002	0.002	0.001	0.002
μ	0	0	0	0	0	0	0	0
ν	0.162	0.241	0.201	0.198	0.117	0.151	0.148	0.276

	2014	2015	2016	2017	2018	2019	2020	2021
σ	0.001	0.002	0.002	0.001	0.002	0.002	0.003	0.002
μ	0	0	0	0	0	0	0	0
ν	0.228	0.248	0.097	0.237	0.152	0.149	0.191	0.241

Table 3. Group ABC Parameters

Appendix B. Minimum Volatility Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	0.103	0.133	0.077	0.118	0.165	0.202	0.135	0.217
VBMFX	0.923	0.883	0.929	0.907	0.867	0.874	0.872	0.916
VGSLX	-0.026	-0.015	-0.006	-0.025	-0.033	-0.076	-0.007	-0.133

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	0.157	0.197	0.185	0.263	0.093	0.188	0.084	0.085
VBMFX	0.887	0.907	0.901	0.799	0.955	0.888	0.946	0.906
VGSLX	-0.044	-0.103	-0.086	-0.062	-0.049	-0.076	-0.03	0.01

Table 4. Group A Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	0.222	0.149	0.094	0.119	0.299	0.304	0.159	0.242
VBMFX	0.886	0.876	0.92	0.907	0.845	0.853	0.867	0.918
VGSLX	-0.009	-0.019	-0.018	-0.021	-0.033	-0.061	-0.003	-0.128
VSMAX	-0.091	0.025	0.027	-0.025	-0.039	-0.064	-0.014	-0.003
VEIEX	-0.004	-0.036	-0.001	0.014	-0.007	-0.006	-0.004	-0.028
VGENX	-0.003	0.006	-0.022	0.005	-0.064	-0.027	-0.005	0

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	0.151	0.128	0.197	0.255	0.052	0.124	0.099	0.005
VBMFX	0.885	0.899	0.91	0.803	0.951	0.898	0.946	0.886
VGSLX	-0.048	-0.113	-0.085	-0.066	-0.049	-0.075	-0.027	-0.005
VSMAX	0.003	0.101	0.033	0.002	0.039	0.042	0.006	-0.004
VEIEX	0.015	0.011	-0.063	-0.012	0.008	-0.029	-0.006	0.015
VGENX	-0.007	-0.026	0.009	0.017	-0.001	0.041	-0.018	0.103

Table 5. Group AB Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	0.181	0.147	0.086	0.117	0.286	0.336	0.169	0.241
VBMFX	0.881	0.876	0.919	0.905	0.834	0.849	0.868	0.916
VGSLX	-0.012	-0.019	-0.019	-0.022	-0.036	-0.067	-0.003	-0.129
VSMAX	-0.08	0.024	0.025	-0.026	-0.037	-0.057	-0.015	0.003
VEIEX	-0.002	-0.036	0	0.013	-0.006	-0.005	-0.004	-0.03
VGENX	0.002	0.006	-0.02	0.006	-0.064	-0.038	-0.006	0.001
AMZN	-0.009	0	0.002	0.006	-0.004	-0.014	-0.005	-0.007
COST	-0.006	0.002	0.014	0.002	0.023	-0.035	-0.001	0.004
WMT	0.046	0	-0.008	0	0.003	0.031	-0.002	0.002

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	0.122	0.121	0.21	0.298	0.085	0.14	0.051	0.055
VBMFX	0.878	0.893	0.908	0.794	0.943	0.885	0.927	0.878
VGSLX	-0.053	-0.121	-0.085	-0.068	-0.055	-0.078	-0.034	-0.005
VSMAX	0.022	0.098	0.027	-0.015	0.045	0.042	0.024	-0.018
VEIEX	0.016	0.014	-0.062	-0.012	0.01	-0.025	0.006	0.019
VGENX	-0.002	-0.026	0.009	0.018	-0.007	0.039	-0.011	0.085
AMZN	-0.008	-0.013	-0.007	-0.02	-0.017	-0.019	-0.02	-0.025
COST	0.014	0.019	0.003	-0.013	-0.008	-0.01	0.038	-0.019
WMT	0.012	0.014	-0.002	0.018	0.004	0.027	0.019	0.031

Table 6. Group ABC Allocations

Appendix C. Two-Risk Free Assets Efficient Frontier Parameters and Metrics

	2006	2007	2008	2009	2010	2011	2012	2013
μ_{si}	1.98e-04	1.37e-04	1.42e-05	1.78e-05	1.07e-05	4.36e-06	6.00e-06	4.76e-06
ν_{si}	1.12e-01	1.12e-01	8.52e-02	1.17e-01	1.46e-01	1.43e-01	1.74e-01	2.16e-01
$\omega_{\text{si}}^{\text{tg}}$	9.98e-01	6.20e-01	9.51e-01	1.61e-01	1.46e-01	8.06e-02	1.09e-01	8.61e-01
$\omega_{\text{si}}^{\text{as}}$	9.99e-01	7.87e-01	9.75e-01	4.01e-01	3.83e-01	2.84e-01	3.30e-01	9.28e-01
μ_{cl}	3.10e-04	2.51e-04	1.31e-04	1.35e-04	1.28e-04	1.22e-04	1.24e-04	1.22e-04
ν_{cl}	1.27e-01	8.93e-02	8.43e-02	7.71e-02	9.39e-02	8.40e-02	9.32e-02	2.01e-01
$\omega_{\text{cl}}^{\text{tg}}$	7.78e-01	9.70e-01	9.70e-01	3.67e-01	3.52e-01	2.35e-01	3.80e-01	9.95e-01
$\omega_{\text{cl}}^{\text{as}}$	8.82e-01	9.85e-01	9.85e-01	6.06e-01	5.93e-01	4.85e-01	6.16e-01	9.98e-01

	2014	2015	2016	2017	2018	2019	2020	2021
μ_{si}	8.72e-06	2.45e-05	3.28e-05	6.87e-05	1.03e-04	6.26e-05	3.95e-06	1.54e-05
ν_{si}	2.02e-01	1.32e-02	7.56e-02	2.17e-01	7.04e-02	2.39e-01	1.40e-01	1.53e-01
$\omega_{\text{si}}^{\text{tg}}$	3.62e-01	9.12e-01	2.97e-01	3.85e-01	7.13e-02	1.83e-01	2.39e-01	9.95e-01
$\omega_{\text{si}}^{\text{as}}$	6.02e-01	9.55e-01	5.45e-01	6.21e-01	2.67e-01	4.28e-01	4.89e-01	9.98e-01
μ_{cl}	1.26e-04	1.41e-04	1.49e-04	1.85e-04	2.18e-04	1.78e-04	1.21e-04	1.32e-04
ν_{cl}	1.44e-01	6.41e-02	4.12e-02	1.60e-01	1.41e-01	1.79e-01	1.06e-01	1.58e-01
$\omega_{\text{cl}}^{\text{tg}}$	7.10e-01	3.89e-02	9.99e-01	7.08e-01	1.79e-02	3.27e-01	4.17e-01	9.32e-01
$\omega_{\text{cl}}^{\text{as}}$	8.43e-01	1.97e-01	9.99e-01	8.41e-01	1.34e-01	5.72e-01	6.46e-01	9.65e-01

Table 7. Group A Parameters and Metrics

	2006	2007	2008	2009	2010	2011	2012	2013
μ_{si}	1.98e-04	1.37e-04	1.42e-05	1.78e-05	1.07e-05	4.36e-06	6.00e-06	4.76e-06
ν_{si}	1.46e-01	1.87e-01	1.23e-01	1.97e-01	1.64e-01	2.04e-01	2.09e-01	2.92e-01
$\omega_{\text{si}}^{\text{tg}}$	9.99e-01	9.21e-01	9.89e-01	6.69e-01	3.57e-01	4.64e-01	3.64e-01	8.89e-01
$\omega_{\text{si}}^{\text{as}}$	9.99e-01	9.60e-01	9.94e-01	8.18e-01	5.97e-01	6.81e-01	6.03e-01	9.43e-01
μ_{cl}	3.10e-04	2.51e-04	1.31e-04	1.35e-04	1.28e-04	1.22e-04	1.24e-04	1.22e-04
ν_{cl}	1.54e-01	1.80e-01	1.24e-01	1.75e-01	1.21e-01	1.62e-01	1.47e-01	2.77e-01
$\omega_{\text{cl}}^{\text{tg}}$	8.92e-01	1	9.73e-01	8.50e-01	6.54e-01	7.33e-01	7.35e-01	9.88e-01
$\omega_{\text{cl}}^{\text{as}}$	9.44e-01	1	9.86e-01	9.22e-01	8.08e-01	8.56e-01	8.58e-01	9.94e-01

	2014	2015	2016	2017	2018	2019	2020	2021
μ_{si}	8.72e-06	2.45e-05	3.28e-05	6.87e-05	1.03e-04	6.26e-05	3.95e-06	1.54e-05
ν_{si}	2.52e-01	1.13e-01	1.08e-01	2.63e-01	1.28e-01	2.43e-01	2.27e-01	1.94e-01
$\omega_{\text{si}}^{\text{tg}}$	5.95e-01	9.98e-01	5.62e-01	6.31e-01	6.54e-01	2.77e-01	6.51e-01	9.96e-01
$\omega_{\text{si}}^{\text{as}}$	7.72e-01	9.99e-01	7.50e-01	7.94e-01	8.09e-01	5.26e-01	8.07e-01	9.98e-01
μ_{cl}	1.26e-04	1.41e-04	1.49e-04	1.85e-04	2.18e-04	1.78e-04	1.21e-04	1.32e-04
ν_{cl}	2.09e-01	1.31e-01	8.11e-02	2.22e-01	1.80e-01	1.86e-01	2.06e-01	1.99e-01
$\omega_{\text{cl}}^{\text{tg}}$	8.66e-01	7.48e-01	9.97e-01	8.84e-01	3.31e-01	4.70e-01	7.95e-01	9.51e-01
$\omega_{\text{cl}}^{\text{as}}$	9.31e-01	8.65e-01	9.98e-01	9.40e-01	5.75e-01	6.86e-01	8.92e-01	9.75e-01

Table 8. Group AB Parameters and Metrics

	2006	2007	2008	2009	2010	2011	2012	2013
μ_{si}	1.98e-04	1.37e-04	1.42e-05	1.78e-05	1.07e-05	4.36e-06	6.00e-06	4.76e-06
ν_{si}	1.62e-01	2.47e-01	2.01e-01	2.32e-01	1.79e-01	2.13e-01	2.20e-01	2.92e-01
$\omega_{\text{si}}^{\text{tg}}$	1	9.51e-01	9.97e-01	7.32e-01	4.24e-01	5.07e-01	4.53e-01	8.91e-01
$\omega_{\text{si}}^{\text{as}}$	1	9.75e-01	9.99e-01	8.56e-01	6.51e-01	7.12e-01	6.73e-01	9.44e-01
μ_{cl}	3.10e-04	2.51e-04	1.31e-04	1.35e-04	1.28e-04	1.22e-04	1.24e-04	1.22e-04
ν_{cl}	1.72e-01	2.41e-01	2.02e-01	2.12e-01	1.39e-01	1.72e-01	1.64e-01	2.78e-01
$\omega_{\text{cl}}^{\text{tg}}$	8.85e-01	1	9.86e-01	8.77e-01	7.07e-01	7.76e-01	8.11e-01	9.89e-01
$\omega_{\text{cl}}^{\text{as}}$	9.41e-01	1	9.93e-01	9.36e-01	8.41e-01	8.81e-01	9.01e-01	9.94e-01

	2014	2015	2016	2017	2018	2019	2020	2021
μ_{si}	8.72e-06	2.45e-05	3.28e-05	6.87e-05	1.03e-04	6.26e-05	3.95e-06	1.54e-05
ν_{si}	2.81e-01	2.51e-01	1.19e-01	2.89e-01	1.79e-01	2.61e-01	2.34e-01	2.41e-01
$\omega_{\text{si}}^{\text{tg}}$	6.57e-01	9.83e-01	6.57e-01	6.74e-01	7.22e-01	3.25e-01	6.71e-01	9.99e-01
$\omega_{\text{si}}^{\text{as}}$	8.10e-01	9.92e-01	8.11e-01	8.21e-01	8.50e-01	5.70e-01	8.19e-01	9.99e-01
μ_{cl}	1.26e-04	1.41e-04	1.49e-04	1.85e-04	2.18e-04	1.78e-04	1.21e-04	1.32e-04
ν_{cl}	2.41e-01	2.65e-01	9.68e-02	2.50e-01	2.26e-01	2.06e-01	2.12e-01	2.46e-01
$\omega_{\text{cl}}^{\text{tg}}$	8.90e-01	8.76e-01	9.99e-01	9.00e-01	4.53e-01	5.21e-01	8.14e-01	9.58e-01
$\omega_{\text{cl}}^{\text{as}}$	9.44e-01	9.36e-01	1	9.49e-01	6.73e-01	7.22e-01	9.02e-01	9.79e-01

Table 9. Group ABC Parameters and Metrics

Appendix D. Metrics for Assets Compared to Group ABC Markowitz Frontier

	2006	2007	2008	2009	2010	2011	2012	2013
ω^{δ}	0.019	0.035	0.089	0.054	0.04	0.067	0.025	0.026
$\omega^{\rho}_{\text{si}}$	0.608	0.95	1	0.733	0.703	0.942	0.653	0.447
$\omega^{\rho}_{\text{cl}}$	0.734	0.996	1	0.74	0.691	0.975	0.625	0.478
ω^{μ}	0.293	0.498	0.65	0.388	0.369	0.518	0.328	0.243
ω^{σ}	0.865	0.978	0.945	0.964	0.951	0.992	0.927	0.83

	2014	2015	2016	2017	2018	2019	2020	2021
ω^{δ}	0.023	0.04	0.036	0.019	0.041	0.031	0.121	0.027
$\omega^{\rho}_{\text{si}}$	0.741	0.971	0.548	0.406	1	0.488	0.824	0.494
$\omega^{\rho}_{\text{cl}}$	0.767	1	0.589	0.424	1	0.422	0.831	0.561
ω^{μ}	0.408	0.472	0.293	0.236	0.525	0.192	0.436	0.243
ω^{σ}	0.965	0.979	0.89	0.804	0.988	0.77	0.984	0.832

Table 10. Metrics for VFINX

	2006	2007	2008	2009	2010	2011	2012	2013
ω^{δ}	0.005	0.006	0.01	0.007	0.008	0.01	0.006	0.01
$\omega^{\rho}_{\text{si}}$	1	0.781	0.761	0.68	0.485	0.499	0.599	1
$\omega^{\rho}_{\text{cl}}$	1	0.966	0.908	0.839	0.666	0.631	0.872	1
ω^{μ}	0.6	0.47	0.3	0.667	0.567	0.45	0.658	0.886
ω^{σ}	0.366	0.524	0.447	0.447	0.651	0.769	0.641	0.365

	2014	2015	2016	2017	2018	2019	2020	2021
ω^{δ}	0.005	0.007	0.01	0.006	0.005	0.007	0.017	0.007
$\omega^{\rho}_{\text{si}}$	0.568	1	0.746	0.872	1	0.54	0.597	1
$\omega^{\rho}_{\text{cl}}$	0.771	1	1	1	1	0.668	0.734	1
ω^{μ}	0.524	0.437	0.694	0.752	0.349	0.666	0.638	0.661
ω^{σ}	0.628	0.652	0.584	0.595	0.48	0.654	0.447	0.504

Table 11. Metrics for VBMFX

	2006	2007	2008	2009	2010	2011	2012	2013
ω^δ	0.027	0.054	0.193	0.117	0.059	0.09	0.029	0.042
ω^ρ_{si}	0.332	1	1	0.813	0.64	0.876	0.64	0.952
ω^ρ_{cl}	0.442	1	1	0.808	0.583	0.882	0.602	0.992
ω^μ	0.157	0.617	0.546	0.409	0.286	0.457	0.313	0.506
ω^σ	0.711	0.964	0.993	0.981	0.898	0.992	0.917	0.984

	2014	2015	2016	2017	2018	2019	2020	2021
ω^δ	0.033	0.048	0.039	0.021	0.037	0.023	0.18	0.035
ω^ρ_{si}	0.474	0.953	0.729	0.923	1	0.514	0.973	0.393
ω^ρ_{cl}	0.455	0.996	0.778	0.983	1	0.458	0.991	0.455
ω^μ	0.24	0.464	0.39	0.527	0.549	0.218	0.52	0.192
ω^σ	0.838	0.982	0.963	0.976	0.983	0.808	0.994	0.77

TABLE 12. Metrics for VGSLX

Appendix E. Safe Tangent Portfolio Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	0.266	0.448	-0.904	0.188	0.098	0.124	0.15	1.005
VBMFX	6.069	0.832	1.527	0.841	0.856	0.888	0.821	0.491
VGSLX	-5.335	-0.281	0.377	-0.029	0.046	-0.012	0.029	-0.496

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	0.099	0.314	0.309	0.496	0.092	0.233	0.241	1.019
VBMFX	0.774	1.449	0.772	0.63	0.912	0.778	0.909	-2.645
VGSLX	0.127	-0.763	-0.081	-0.126	-0.004	-0.011	-0.149	2.626

Table 13. Group A Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	7.274	-0.363	-0.647	-0.216	-0.024	0.411	0.318	1.309
VBMFX	-6.673	0.987	1.959	0.859	0.863	0.874	0.779	0.636
VGSLX	6.495	-0.208	0.795	-0.035	0.001	0.016	0.018	-0.382
VSMAX	-9.364	-0.339	-0.62	0.094	0.205	-0.124	-0.022	0.249
VEIEX	2.756	0.568	-1.609	0.421	0.087	-0.238	0.076	-0.47
VGENX	0.513	0.355	1.123	-0.123	-0.132	0.062	-0.169	-0.342

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	0.439	-8.218	-0.048	0.663	-0.454	0.335	0.104	3.466
VBMFX	0.76	0.742	0.878	0.5	0.808	0.783	0.861	-2.231
VGSLX	0.143	0.069	-0.087	-0.093	0.049	-0.015	-0.218	2.106
VSMAX	-0.154	3.461	0.245	-0.218	0.266	-0.051	0.416	-2.349
VEIEX	-0.079	3.312	-0.12	0.218	0.218	-0.039	0.118	-1.083
VGENX	-0.109	1.634	0.132	-0.071	0.114	-0.013	-0.281	1.092

Table 14. Group AB Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	-53.049	-0.963	-5.064	-0.124	-0.025	0.272	0.243	1.328
VBMFX	39.971	0.974	1.381	0.885	0.847	0.845	0.768	0.636
VGSLX	-31.784	-0.121	1.397	-0.048	-0.018	0.002	0.014	-0.383
VSMAX	40.967	-0.504	-0.878	0.057	0.166	-0.077	-0.015	0.241
VEIEX	-12.571	0.523	-1.677	0.392	0.087	-0.235	0.085	-0.469
VGENX	2.115	0.483	2.236	-0.146	-0.122	0.092	-0.165	-0.346
AMZN	3.711	0.159	0.383	0.082	0.024	-0.003	0.019	0.01
COST	0.714	0.355	-1.371	-0.005	0.11	0.014	0.047	-0.026
WMT	10.927	0.096	4.594	-0.092	-0.068	0.089	0.003	0.009

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	0.549	-0.489	-0.175	0.53	-0.05	0.318	-0.084	5.192
VBMFX	0.763	0.819	0.859	0.505	0.789	0.75	0.833	-3.827
VGSLX	0.15	-0.203	-0.102	-0.118	-0.008	-0.026	-0.189	3.404
VSMAX	-0.15	0.384	0.306	-0.184	0.261	-0.037	0.456	-3.569
VEIEX	-0.057	0.612	-0.108	0.205	0.183	-0.019	0.105	-1.01
VGENX	-0.143	0.202	0.134	-0.047	0.049	-0.01	-0.242	2.1
AMZN	-0.049	-0.498	0.027	0.026	-0.135	-0.053	0.05	-1.691
COST	0.045	-0.63	-0.053	-0.007	-0.085	0.045	0.04	3.459
WMT	-0.108	0.802	0.112	0.09	-0.004	0.032	0.029	-3.057

TABLE 15. Group ABC Allocations

Appendix F. Credit Tangent Portfolio Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	0.116	1.53	1.343	0.239	0.046	0.057	0.169	4.793
VBMFX	1.336	0.658	0.158	0.793	0.848	0.899	0.757	-1.552
VGSLX	-0.452	-1.189	-0.501	-0.032	0.107	0.044	0.074	-2.241

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	0.035	0.204	-4.917	0.722	0.093	0.254	0.32	-0.16
VBMFX	0.653	0.94	6.206	0.466	0.934	0.727	0.89	1.834
VGSLX	0.311	-0.144	-0.289	-0.188	-0.027	0.019	-0.21	-0.674

Table 16. Group A Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	-0.493	23.09	0.565	-0.442	-0.296	0.495	0.508	3.648
VBMFX	1.652	-4.081	0.259	0.826	0.879	0.89	0.674	0.017
VGSLX	-0.668	8.412	-0.535	-0.044	0.029	0.076	0.044	-0.938
VSMAX	0.848	16.284	0.439	0.174	0.412	-0.171	-0.032	0.802
VEIEX	-0.284	-27.08	1.022	0.695	0.166	-0.42	0.173	-1.437
VGENX	-0.055	-15.625	-0.749	-0.209	-0.19	0.131	-0.367	-1.092

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	0.755	-0.573	-3.693	1.114	-0.207	0.445	0.106	-1.02
VBMFX	0.623	0.886	0.403	0.164	0.878	0.723	0.823	1.809
VGSLX	0.351	-0.098	-0.11	-0.123	0.001	0.017	-0.302	-0.629
VSMAX	-0.326	0.384	3.397	-0.461	0.155	-0.099	0.598	0.691
VEIEX	-0.181	0.288	-0.957	0.474	0.115	-0.044	0.173	0.34
VGENX	-0.222	0.113	1.961	-0.167	0.058	-0.041	-0.398	-0.191

Table 17. Group AB Allocations

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	-0.76	-92.3	2.316	-0.271	-0.276	0.219	0.337	3.786
VBMFX	1.572	9.054	0.719	0.873	0.857	0.842	0.641	0.002
VGSLX	-0.574	-8.553	-0.631	-0.064	-0.004	0.06	0.036	-0.956
VSMAX	0.645	-43.93	0.415	0.108	0.33	-0.095	-0.015	0.78
VEIEX	-0.225	46.497	0.725	0.624	0.161	-0.427	0.2	-1.463
VGENX	0.039	39.689	-0.996	-0.239	-0.17	0.201	-0.368	-1.131
AMZN	0.057	13.166	-0.163	0.128	0.046	0.006	0.049	0.047
COST	0.007	29.389	0.614	-0.01	0.181	0.055	0.109	-0.093
WMT	0.238	7.989	-1.999	-0.149	-0.126	0.138	0.011	0.026

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	1.002	-0.089	-9.453	0.782	0.009	0.407	-0.147	-0.743
VBMFX	0.642	0.868	-0.309	0.192	0.856	0.683	0.789	1.609
VGSLX	0.367	-0.149	-0.517	-0.173	-0.028	0	-0.261	-0.535
VSMAX	-0.333	0.197	7.027	-0.368	0.167	-0.076	0.657	0.534
VEIEX	-0.134	0.221	-1.208	0.441	0.107	-0.017	0.152	0.179
VGENX	-0.293	0.053	3.147	-0.117	0.025	-0.035	-0.349	-0.228
AMZN	-0.091	-0.181	0.853	0.075	-0.084	-0.07	0.083	0.234
COST	0.077	-0.205	-1.385	-0.002	-0.051	0.074	0.04	-0.56
WMT	-0.236	0.286	2.844	0.169	-0.001	0.034	0.034	0.51

Table 18. Group ABC Allocations

Appendix G. Long Tangent Allocations and Metrics

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	0	1.04e-01	0	1.35e-01	1.05e-01	1.11e-01	1.46e-01	01
VBMFX	0	8.87e-01	1	8.65e-01	8.57e-01	8.88e-01	8.19e-01	00
VGSLX	1	9.00e-03	0	00	3.80e-02	1.00e-03	3.50e-02	00
μ_{si}	1.98e-04	1.37e-04	1.42e-05	1.78e-05	1.07e-05	4.36e-06	6.00e-06	4.76e-06
μ_{mn}	1.70e-04	-6.48e-04	-1.49e-03	2.34e-04	2.46e-04	1.84e-04	1.61e-04	-8.79e-05
μ_{mx}	1.19e-03	2.71e-04	2.04e-04	1.86e-03	1.15e-03	5.07e-04	6.89e-04	1.13e-03
σ_{mx}	9.16e-03	1.61e-02	2.58e-02	4.24e-02	1.77e-02	1.90e-02	8.62e-03	6.97e-03
μ_{ELT}	1.19e-03	2.62e-04	2.04e-04	3.49e-04	3.19e-04	2.82e-04	2.46e-04	1.13e-03
σ_{ELT}	9.16e-03	2.15e-03	3.95e-03	2.92e-03	2.12e-03	1.94e-03	1.38e-03	6.97e-03
ω^{δ}	2.72e-02	7.74e-03	9.92e-03	1.09e-02	7.36e-03	6.84e-03	5.03e-03	2.60e-02
$\omega^{\rho}_{\text{si}}$	3.64e-02	4.79e-01	4.36e-01	2.74e-02	5.93e-04	1.58e-03	2.50e-04	2.51e-01
$\omega^{\rho}_{\text{cl}}$	2.46e-01	9.42e-01	7.80e-01	4.96e-02	3.81e-02	1.39e-02	5.00e-02	2.78e-01
ω^{σ}	4.13e-02	1.37e-05	2.81e-01	2.31e-01	1.88e-04	1.98e-02	1.95e-02	6.62e-01
ω^{μ}	4.49e-04	1	9.89e-02	6.45e-02	7.22e-08	8.78e-04	8.23e-04	1.35e-01

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	9.90e-02	0	2.72e-01	4.65e-01	0	2.27e-01	8.80e-02	2.84e-01
VBMFX	7.75e-01	0	7.27e-01	5.35e-01	0	7.67e-01	9.12e-01	1.00e-03
VGSLX	1.26e-01	1	0	00	1	6.00e-03	0	7.15e-01
μ_{si}	8.72e-06	2.45e-05	3.28e-05	6.87e-05	1.03e-04	6.26e-05	3.95e-06	1.54e-05
μ_{mn}	2.24e-04	1.49e-05	1.01e-04	1.38e-04	-1.94e-04	3.31e-04	1.73e-04	-6.72e-05
μ_{mx}	1.08e-03	1.52e-04	4.78e-04	7.90e-04	-3.16e-06	1.11e-03	8.98e-04	1.39e-03
σ_{mx}	7.24e-03	1.08e-02	8.25e-03	4.21e-03	1.02e-02	7.85e-03	2.67e-02	9.38e-03
μ_{ELT}	3.62e-04	1.52e-04	2.03e-04	4.41e-04	-1.84e-04	5.13e-04	3.49e-04	1.29e-03
σ_{ELT}	1.75e-03	1.08e-02	2.36e-03	1.87e-03	1.02e-02	1.89e-03	2.98e-03	8.47e-03
ω^{δ}	7.08e-03	4.75e-02	1.02e-02	6.59e-03	3.73e-02	5.55e-03	1.96e-02	2.95e-02
$\omega^{\rho}_{\text{si}}$	2.31e-06	1.15e-01	4.49e-02	8.11e-02	1	1.73e-03	1.72e-01	1.99e-02
$\omega^{\rho}_{\text{cl}}$	6.36e-02	9.85e-01	4.43e-01	1.42e-01	1	9.69e-03	2.77e-01	1.39e-01
ω^{σ}	5.58e-04	7.42e-04	2.86e-01	3.93e-01	6.22e-02	4.75e-02	3.03e-01	3.17e-02
ω^{μ}	2.17e-07	1.43e-07	5.07e-02	9.51e-02	9.99e-01	2.65e-03	5.12e-01	2.70e-04

Table 19. Group A Allocations and Metrics

	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	0	00	0	00	1.00e-03	8.40e-02	1.36e-01	9.19e-01
VBMFX	2.00e-03	7.95e-01	1	8.48e-01	8.70e-01	8.95e-01	8.18e-01	00
VGSLX	7.57e-01	0	00	0	6.00e-03	2.10e-02	3.10e-02	00
VSMAX	0	00	0	00	1.23e-01	0	1.00e-03	8.10e-02
VEIEX	2.41e-01	7.50e-02	0	1.52e-01	1.00e-03	0	1.40e-02	00
VGENX	0	1.29e-01	0	00	0	00	0	00
μ_{si}	1.98e-04	1.37e-04	1.42e-05	1.78e-05	1.07e-05	4.36e-06	6.00e-06	4.76e-06
μ_{mn}	1.70e-04	-6.48e-04	-2.57e-03	2.34e-04	2.46e-04	-6.90e-04	1.61e-04	-1.72e-04
μ_{mx}	1.19e-03	1.41e-03	2.04e-04	2.48e-03	1.15e-03	5.07e-04	7.37e-04	1.31e-03
σ_{mx}	9.16e-03	1.61e-02	3.21e-02	2.05e-02	1.77e-02	1.90e-02	1.03e-02	9.00e-03
μ_{ELT}	1.17e-03	4.96e-04	2.04e-04	5.74e-04	3.56e-04	2.89e-04	2.48e-04	1.15e-03
σ_{ELT}	8.57e-03	2.95e-03	3.95e-03	3.36e-03	2.22e-03	2.01e-03	1.39e-03	7.05e-03
ω^{δ}	2.27e-02	9.77e-03	9.92e-03	1.27e-02	7.17e-03	7.36e-03	5.10e-03	2.61e-02
$\omega^{\rho}_{\text{si}}$	2.26e-01	3.48e-01	6.11e-01	1.59e-01	5.55e-02	3.03e-01	1.67e-01	4.46e-01
$\omega^{\rho}_{\text{cl}}$	3.53e-01	5.36e-01	8.51e-01	2.51e-01	1.55e-01	4.84e-01	3.92e-01	4.76e-01
ω^{σ}	5.98e-01	5.13e-01	3.81e-01	4.20e-01	2.68e-01	4.48e-01	3.32e-01	8.30e-01
ω^{μ}	1.06e-01	1.63e-01	1.92e-01	1.20e-01	8.09e-02	4.28e-01	2.94e-01	2.42e-01

	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	8.70e-02	0	00	3.15e-01	0	2.21e-01	7.70e-02	1.95e-01
VBMFX	7.53e-01	0	7.75e-01	4.85e-01	0	7.77e-01	9.21e-01	00
VGSLX	1.60e-01	1	0	00	0	1.00e-03	0	6.36e-01
VSMAX	0	00	9.30e-02	0	00	0	1.00e-03	00
VEIEX	0	00	0	2.00e-01	0	00	1.00e-03	00
VGENX	0	00	1.32e-01	0	1	1.00e-03	0	1.68e-01
μ_{si}	8.72e-06	2.45e-05	3.28e-05	6.87e-05	1.03e-04	6.26e-05	3.95e-06	1.54e-05
μ_{mn}	-5.49e-04	-8.33e-04	1.01e-04	1.38e-04	-6.65e-04	3.31e-04	-9.91e-04	-6.72e-05
μ_{mx}	1.08e-03	1.52e-04	1.26e-03	1.10e-03	-3.16e-06	1.11e-03	1.01e-03	1.39e-03
σ_{mx}	1.13e-02	1.60e-02	1.59e-02	5.55e-03	1.32e-02	7.85e-03	3.02e-02	9.38e-03
μ_{ELT}	3.87e-04	1.52e-04	3.11e-04	5.35e-04	-6.58e-04	5.05e-04	3.43e-04	1.26e-03
σ_{ELT}	1.88e-03	1.08e-02	2.87e-03	2.12e-03	1.32e-02	1.85e-03	2.93e-03	8.15e-03
ω^{δ}	7.60e-03	4.75e-02	1.05e-02	8.24e-03	3.63e-02	5.53e-03	1.93e-02	2.91e-02
$\omega^{\rho}_{\text{si}}$	2.02e-01	8.97e-01	1.02e-01	1.61e-01	1	1.70e-02	4.92e-01	2.14e-01
$\omega^{\rho}_{\text{cl}}$	3.35e-01	9.93e-01	3.04e-01	2.54e-01	1	5.40e-02	6.32e-01	3.04e-01
ω^{σ}	5.20e-01	9.78e-01	4.40e-01	5.14e-01	8.71e-01	1.62e-01	2.54e-01	5.85e-01
ω^{μ}	1.85e-01	4.43e-01	8.32e-02	1.33e-01	7.39e-01	3.02e-02	6.43e-01	1.02e-01

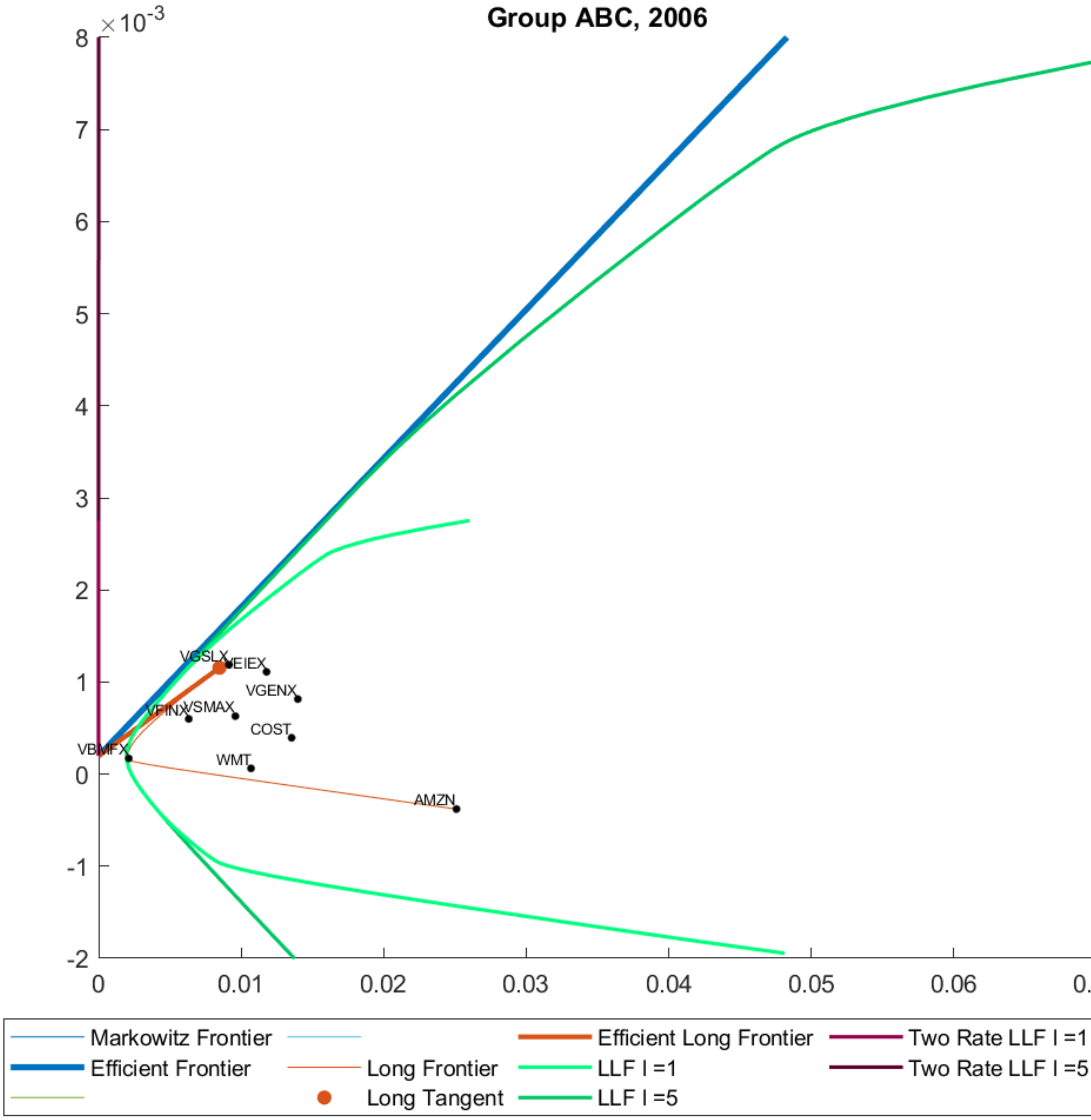
TABLE 20. Group AB Allocations and Metrics

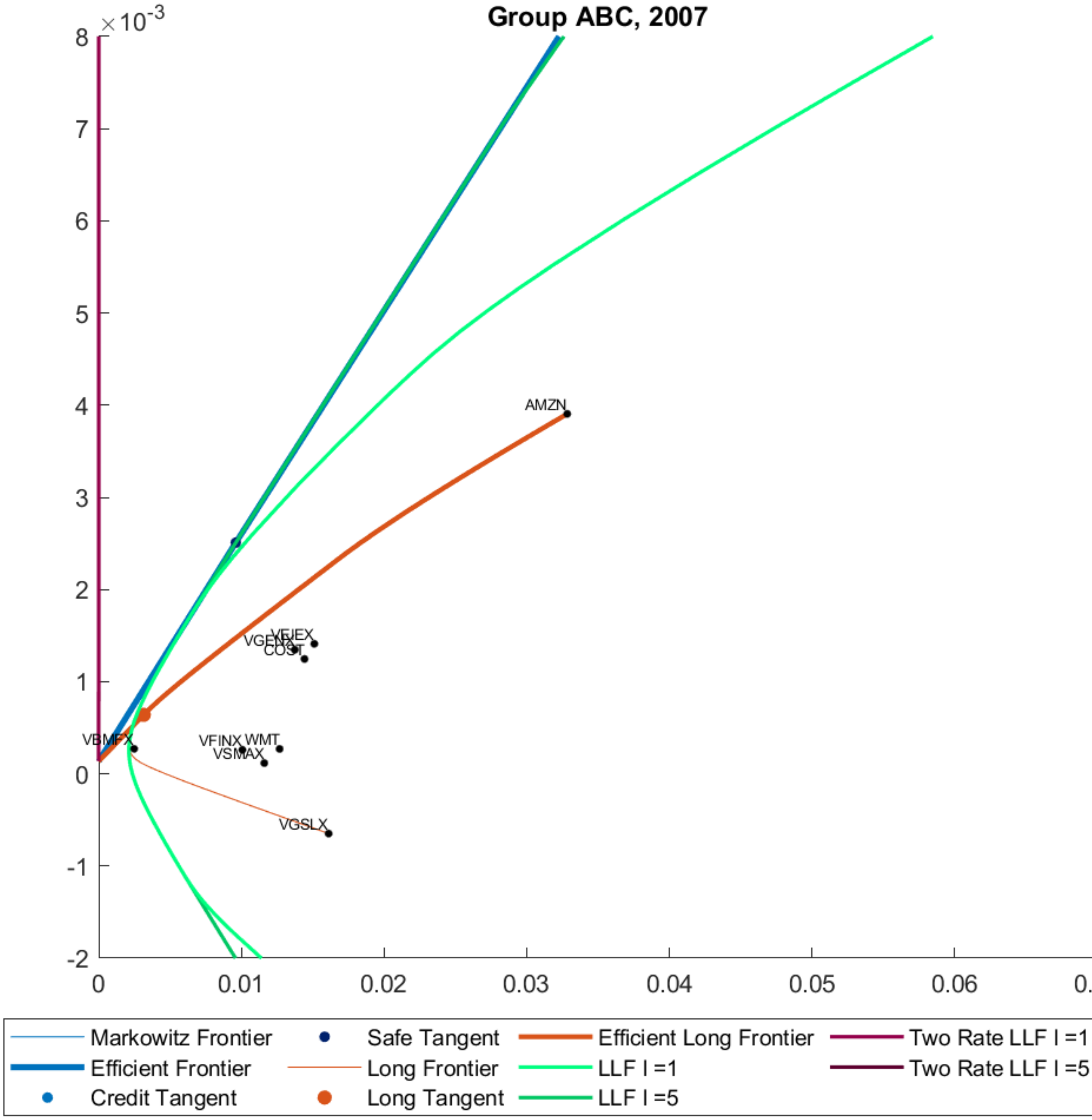
	2006	2007	2008	2009	2010	2011	2012	2013
VFINX	0	00	0	00	0	3.20e-02	6.00e-02	6.19e-01
VBMFX	1.30e-02	7.90e-01	8.66e-01	8.26e-01	8.27e-01	8.36e-01	8.04e-01	00
VGSLX	7.48e-01	0	00	0	00	1.00e-03	2.20e-02	00
VSMAX	0	00	0	00	6.80e-02	0	5.00e-03	1.40e-01
VEIEX	2.39e-01	9.00e-03	0	1.10e-01	4.00e-03	0	2.40e-02	00
VGENX	0	8.20e-02	0	00	0	5.00e-03	0	00
AMZN	0	5.80e-02	0	6.30e-02	2.50e-02	0	2.00e-02	1.16e-01
COST	0	6.10e-02	0	00	7.60e-02	2.60e-02	4.90e-02	00
WMT	0	00	1.34e-01	0	00	9.90e-02	1.70e-02	1.25e-01
μ_{si}	1.98e-04	1.37e-04	1.42e-05	1.78e-05	1.07e-05	4.36e-06	6.00e-06	4.76e-06
μ_{mn}	-3.79e-04	-6.48e-04	-2.57e-03	-5.52e-06	1.64e-04	-6.90e-04	1.61e-04	-1.72e-04
μ_{mx}	1.19e-03	3.91e-03	9.42e-04	4.35e-03	1.37e-03	7.10e-04	1.68e-03	1.98e-03
σ_{mx}	2.51e-02	3.28e-02	3.21e-02	3.31e-02	2.06e-02	1.62e-02	2.00e-02	1.70e-02
μ_{ELT}	1.16e-03	6.40e-04	3.03e-04	7.42e-04	3.83e-04	3.28e-04	2.99e-04	1.20e-03
σ_{ELT}	8.48e-03	3.16e-03	3.93e-03	3.82e-03	2.22e-03	2.04e-03	1.53e-03	7.15e-03
ω^{δ}	2.24e-02	8.10e-03	1.42e-02	1.42e-02	7.07e-03	6.02e-03	4.79e-03	2.46e-02
$\omega^{\rho}_{\text{si}}$	3.01e-01	3.55e-01	6.34e-01	1.82e-01	6.63e-02	2.52e-01	1.30e-01	4.28e-01
$\omega^{\rho}_{\text{cl}}$	4.20e-01	4.88e-01	7.83e-01	2.49e-01	1.74e-01	4.10e-01	3.05e-01	4.56e-01
ω^{σ}	6.75e-01	5.39e-01	3.54e-01	4.99e-01	2.65e-01	4.86e-01	3.92e-01	8.18e-01
ω^{μ}	1.40e-01	1.56e-01	1.64e-01	1.29e-01	7.57e-02	2.98e-01	1.60e-01	2.32e-01

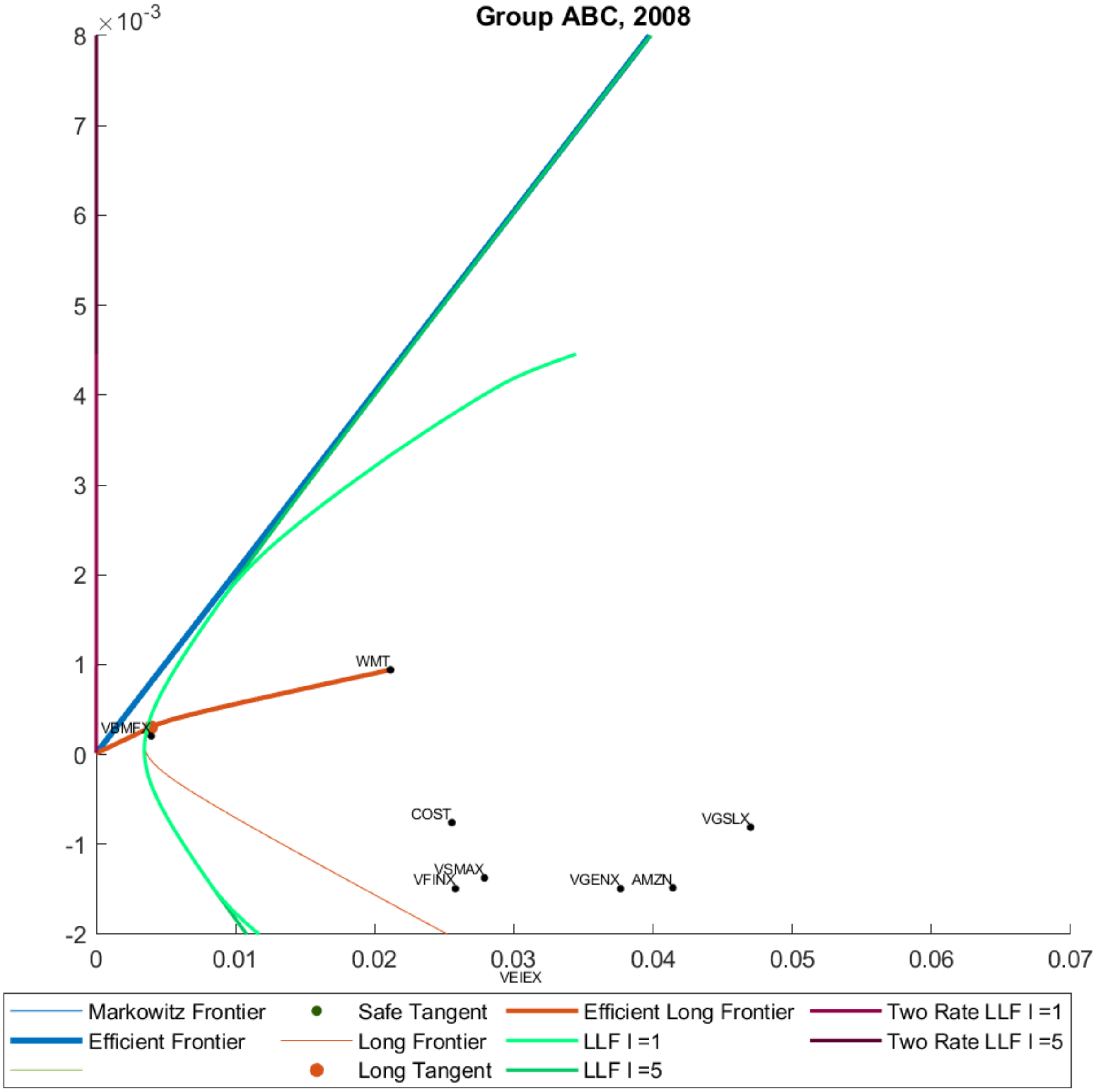
	2014	2015	2016	2017	2018	2019	2020	2021
VFINX	6.20e-02	0	00	1.82e-01	0	1.70e-01	0	1.00e-03
VBMFX	7.57e-01	4.03e-01	7.23e-01	4.41e-01	0	7.42e-01	8.54e-01	00
VGSLX	1.27e-01	0	00	0	00	1.00e-03	0	4.27e-01
VSMAX	0	00	5.90e-02	0	00	0	1.00e-03	00
VEIEX	0	00	0	2.13e-01	0	00	0	00
VGENX	0	00	1.28e-01	0	00	2.00e-03	0	1.80e-01
AMZN	0	5.71e-01	4.00e-03	5.90e-02	8.71e-01	0	8.30e-02	00
COST	5.40e-02	2.60e-02	0	6.00e-03	1.29e-01	5.00e-02	5.10e-02	3.92e-01
WMT	0	00	8.60e-02	9.90e-02	0	3.40e-02	1.10e-02	00
μ_{si}	8.72e-06	2.45e-05	3.28e-05	6.87e-05	1.03e-04	6.26e-05	3.95e-06	1.54e-05
μ_{mn}	-7.84e-04	-1.14e-03	7.19e-05	1.38e-04	-6.65e-04	3.31e-04	-9.91e-04	-6.72e-05
μ_{mx}	1.08e-03	3.31e-03	1.26e-03	1.86e-03	1.26e-03	1.55e-03	2.53e-03	1.73e-03
σ_{mx}	2.03e-02	2.11e-02	1.59e-02	1.32e-02	2.27e-02	1.01e-02	3.02e-02	1.19e-02
μ_{ELT}	3.82e-04	1.91e-03	3.36e-04	7.11e-04	1.16e-03	5.52e-04	5.40e-04	1.46e-03
σ_{ELT}	1.80e-03	1.21e-02	2.94e-03	2.53e-03	2.08e-02	1.96e-03	3.48e-03	7.86e-03
ω^{δ}	7.29e-03	3.89e-02	1.11e-02	9.51e-03	7.39e-02	4.89e-03	1.95e-02	2.47e-02
$\omega^{\rho}_{\text{si}}$	2.61e-01	3.78e-01	1.35e-01	1.21e-01	7.16e-01	4.45e-02	3.40e-01	2.40e-01
$\omega^{\rho}_{\text{cl}}$	4.10e-01	4.49e-01	3.43e-01	1.67e-01	8.00e-01	7.58e-02	4.32e-01	3.16e-01
ω^{σ}	5.31e-01	7.46e-01	4.91e-01	4.74e-01	9.21e-01	2.89e-01	5.61e-01	6.07e-01
ω^{μ}	2.22e-01	1.72e-01	1.03e-01	8.61e-02	3.09e-01	6.83e-02	2.94e-01	1.11e-01

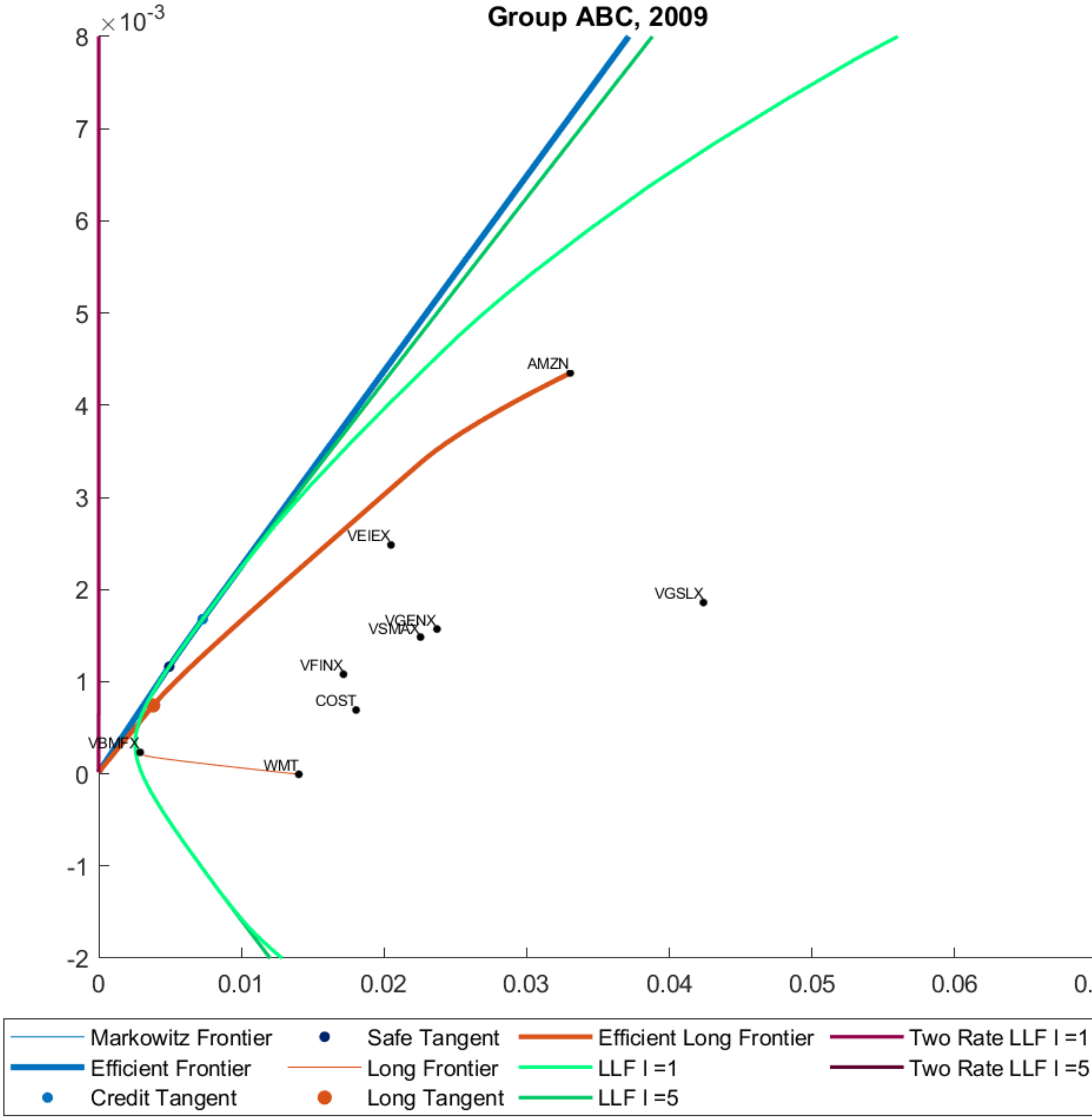
Table 21. Group ABC Allocations and Metrics

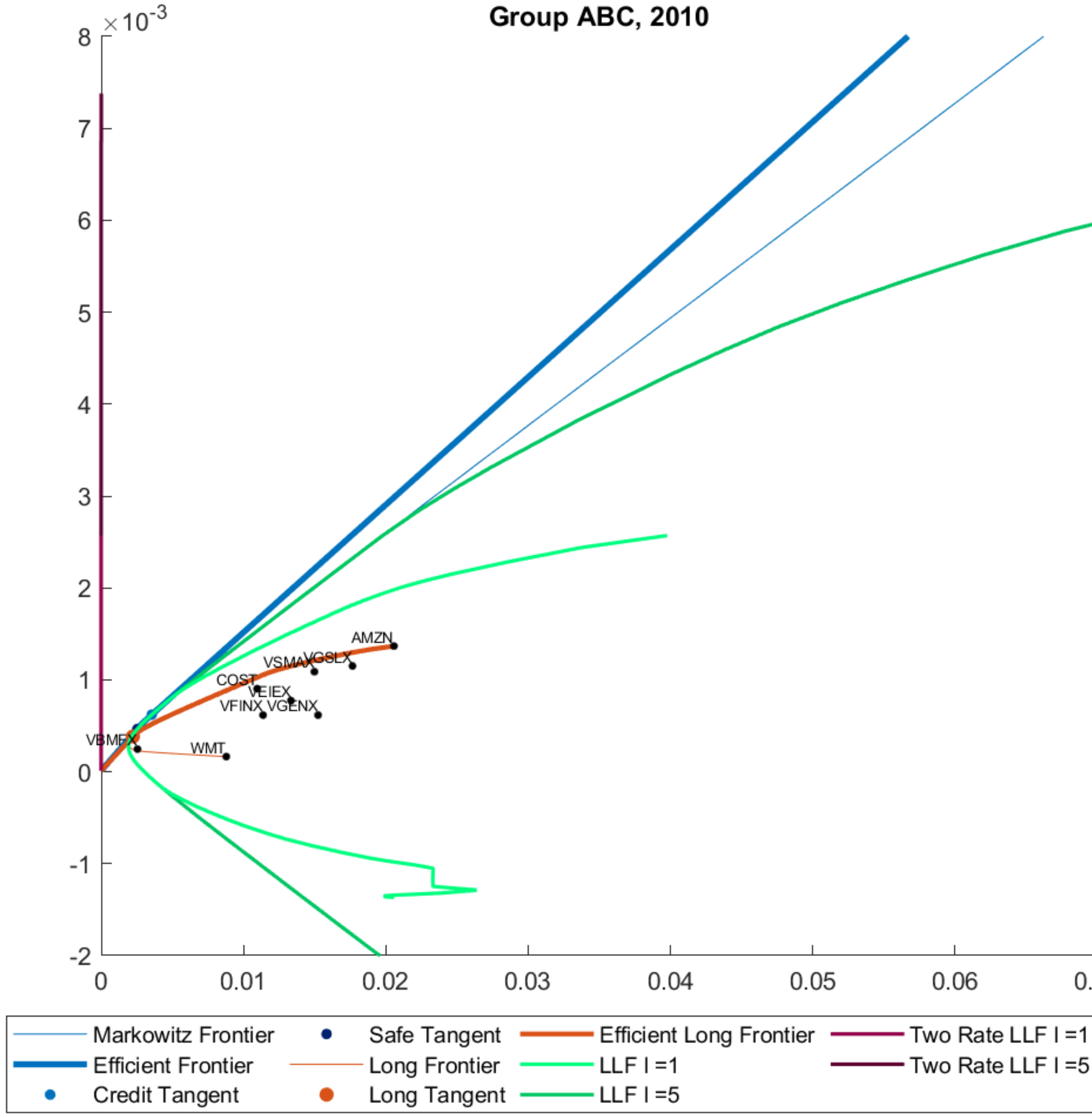
Appendix H. Plots on the $\sigma\mu$ plane

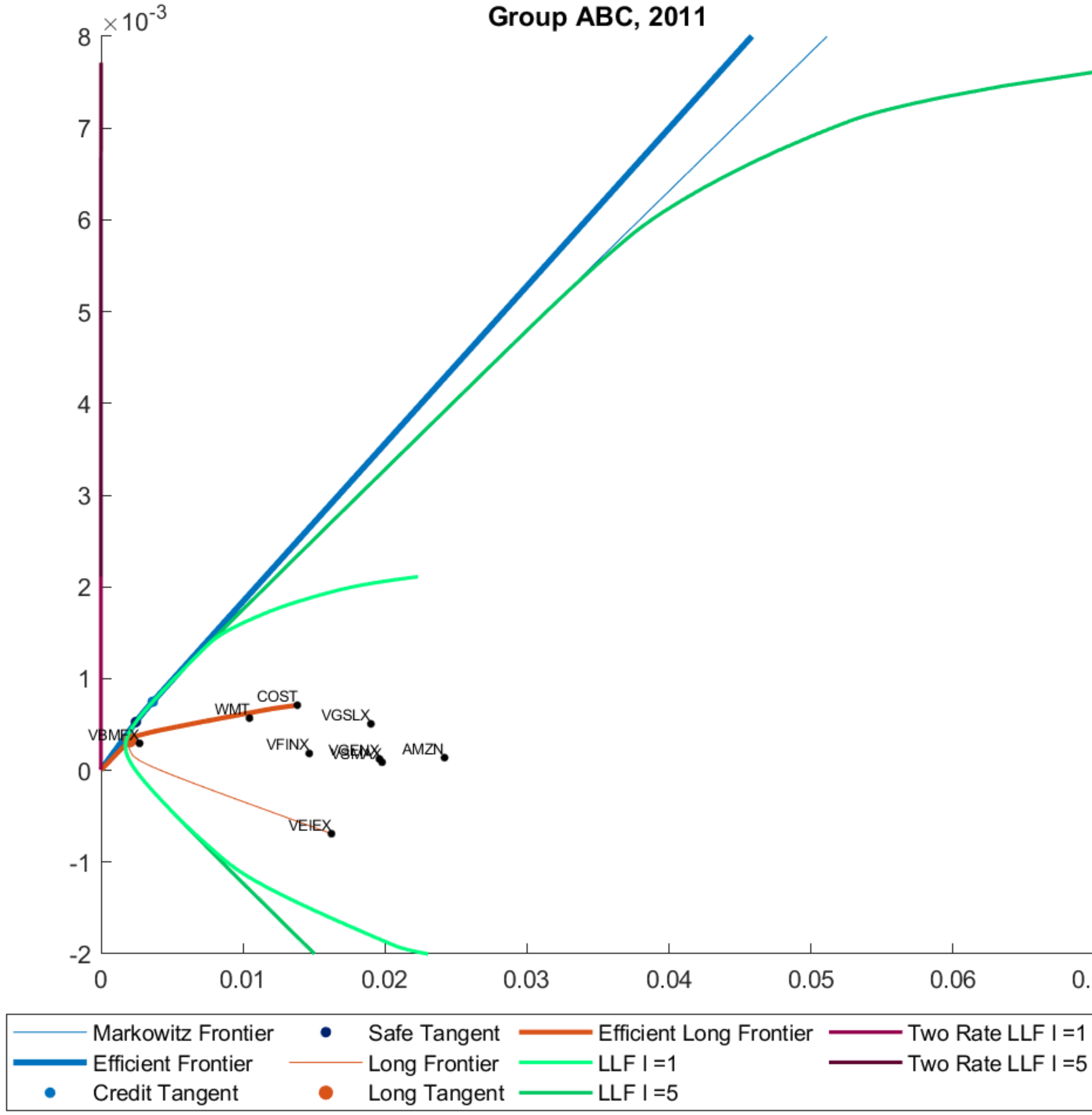


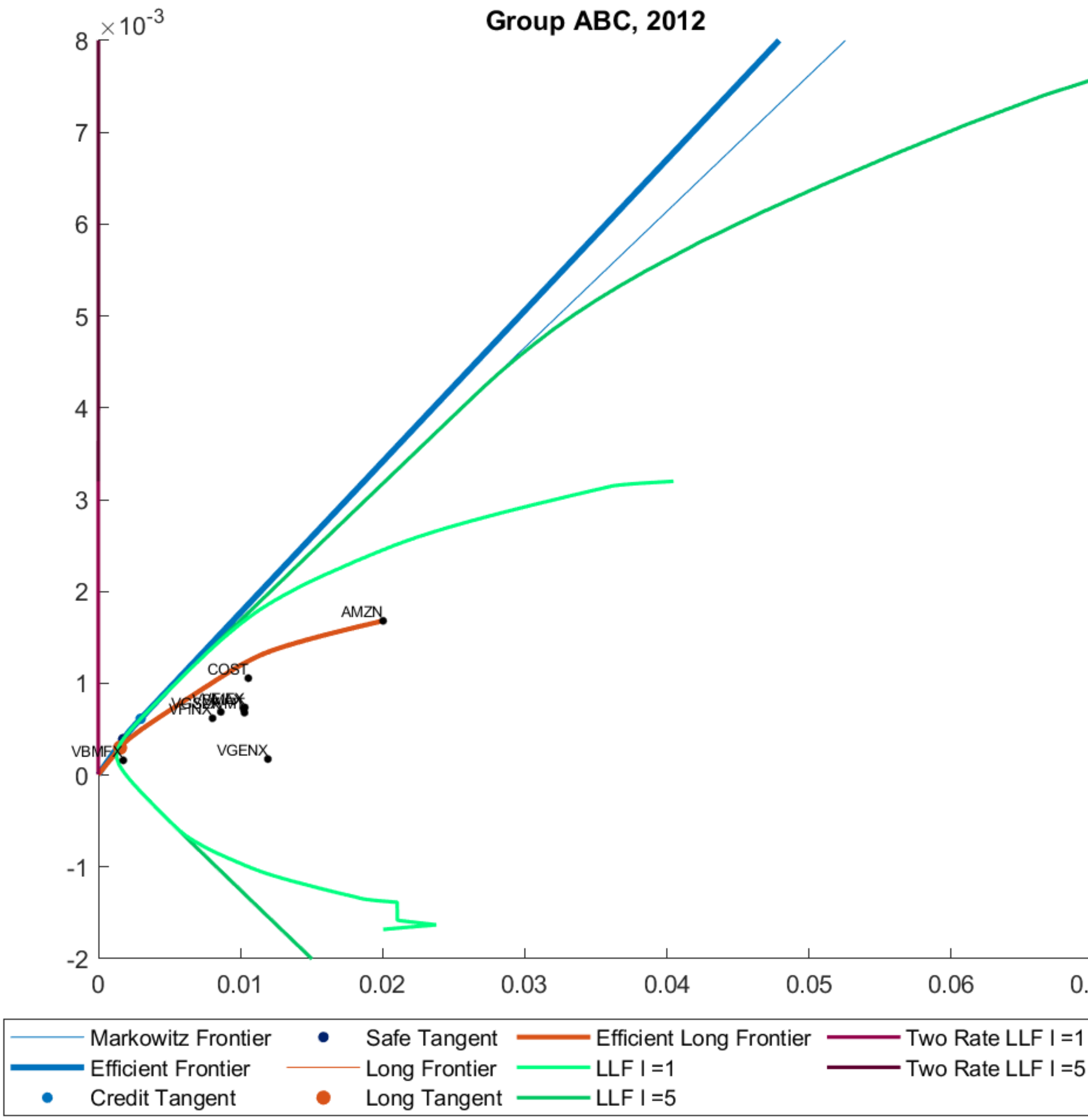

Group ABC, 2007
×10-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
AMZN
VGENX
VFIEX
COST
VBMFX
VFINX
WMT
VSMAX
VGSLX
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

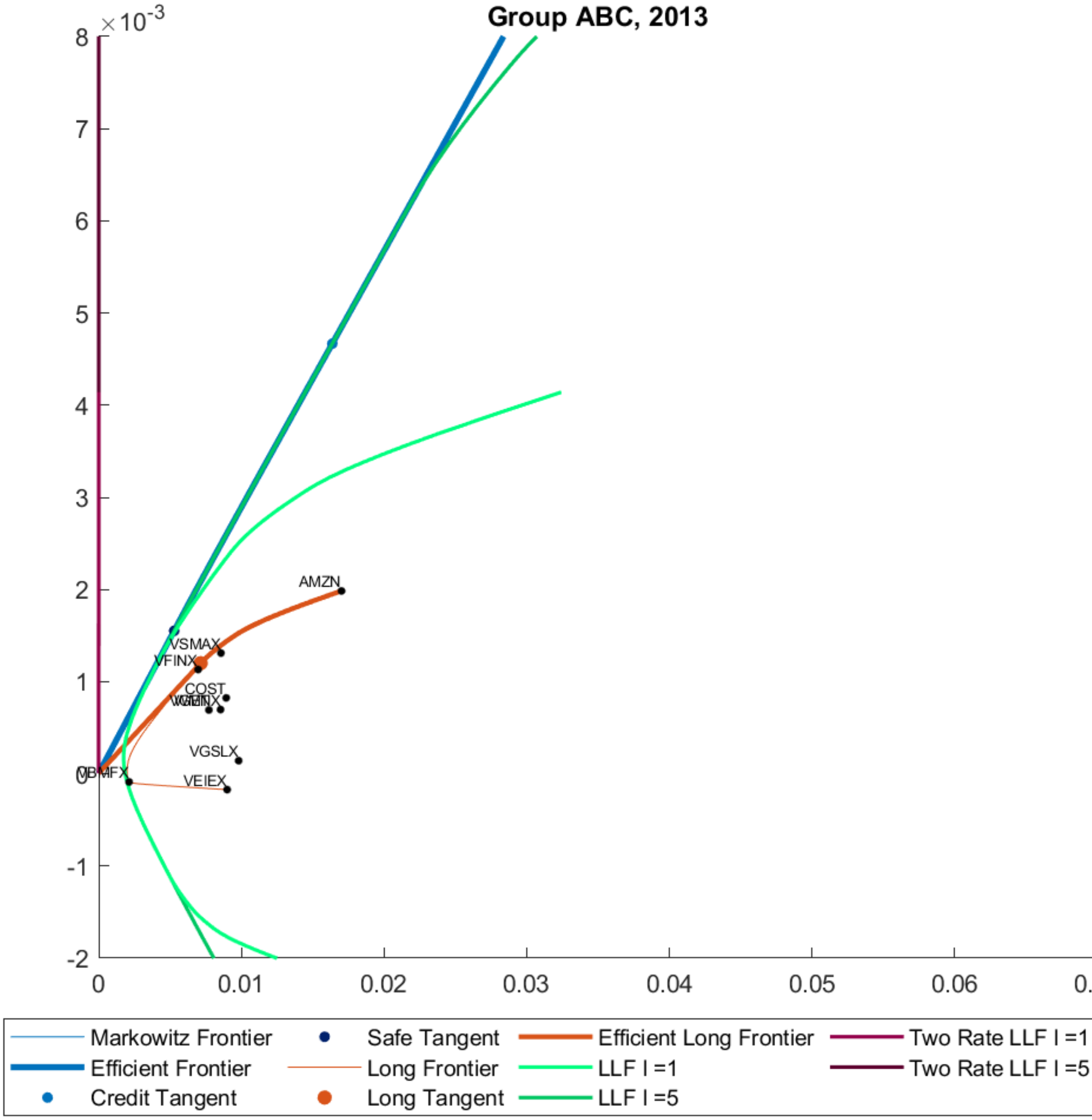

Group ABC, 2008
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
WMT
VBMFX
COST
VGSLX
VSMAX
VFINX
VGENX
AMZN
VEIEX
Markowitz Frontier
Efficient Frontier
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

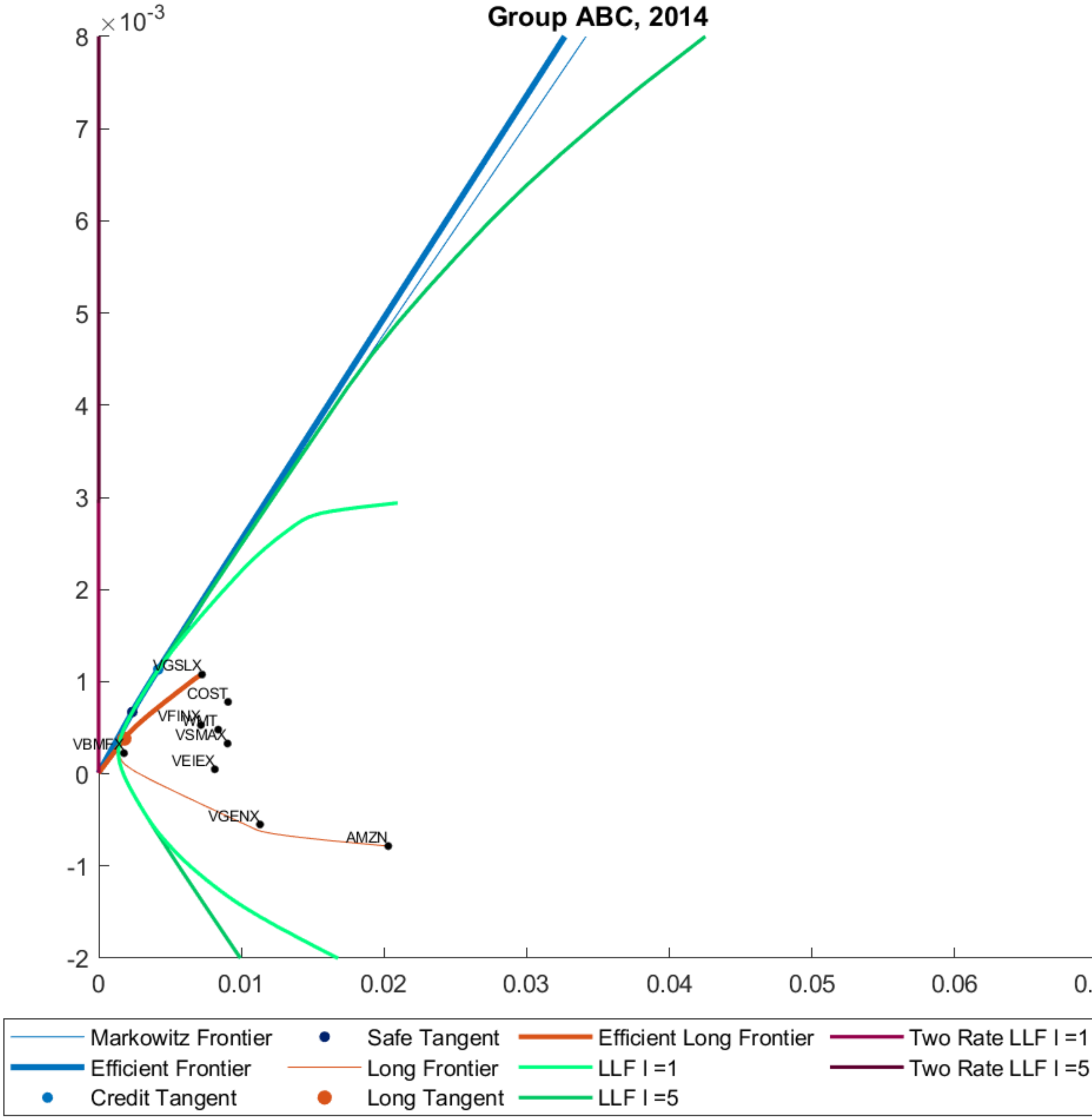

Group ABC, 2009
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
AMZN
VEIEX
VGSLX
VGENX
VSMAX
VFINX
COST
VBMFX
WMT
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

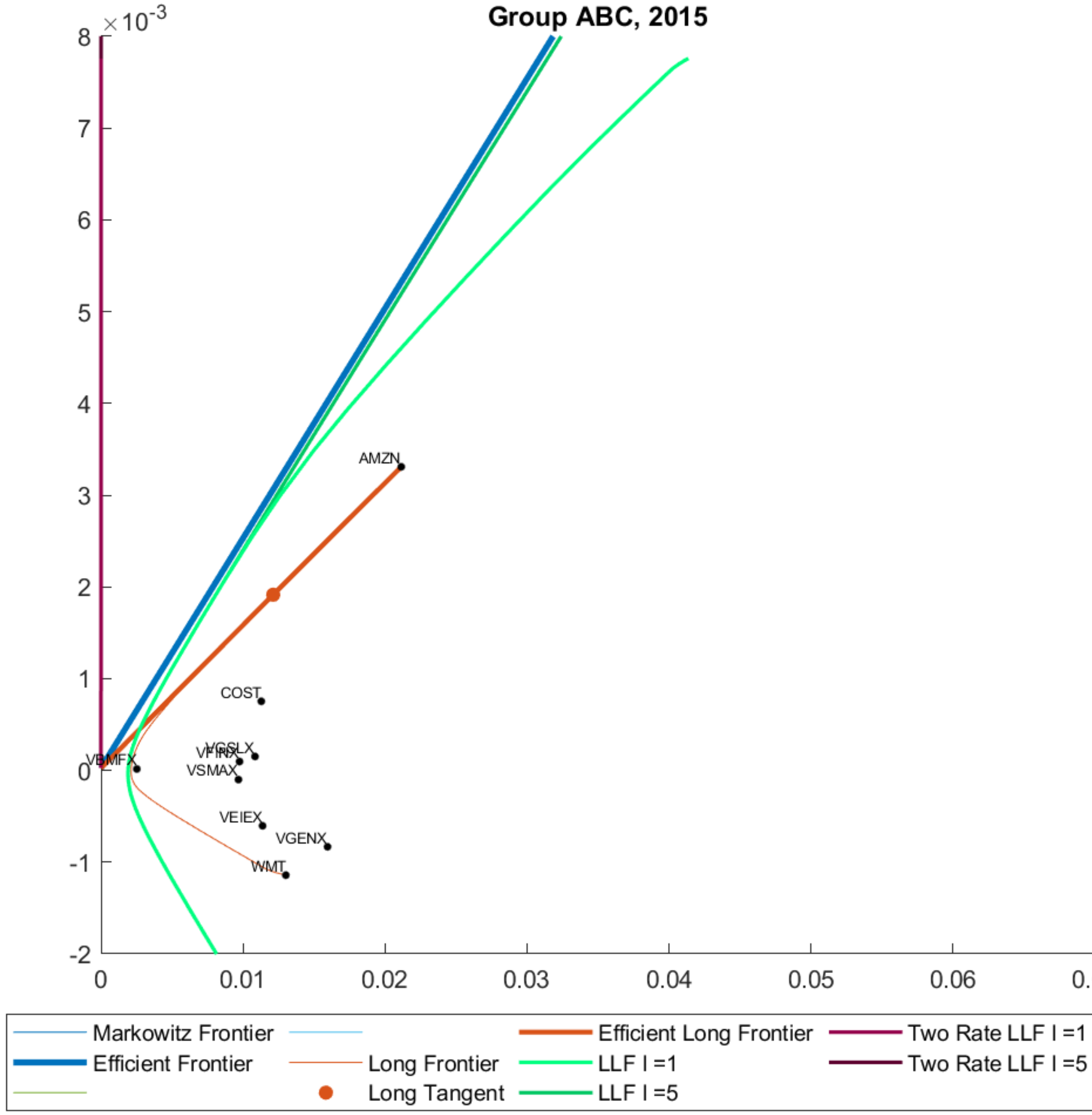

Group ABC, 2010
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
AMZN
VSMAX
VGSLX
COST
VEIEX
VFINX
VGENX
VBMFX
WMT
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

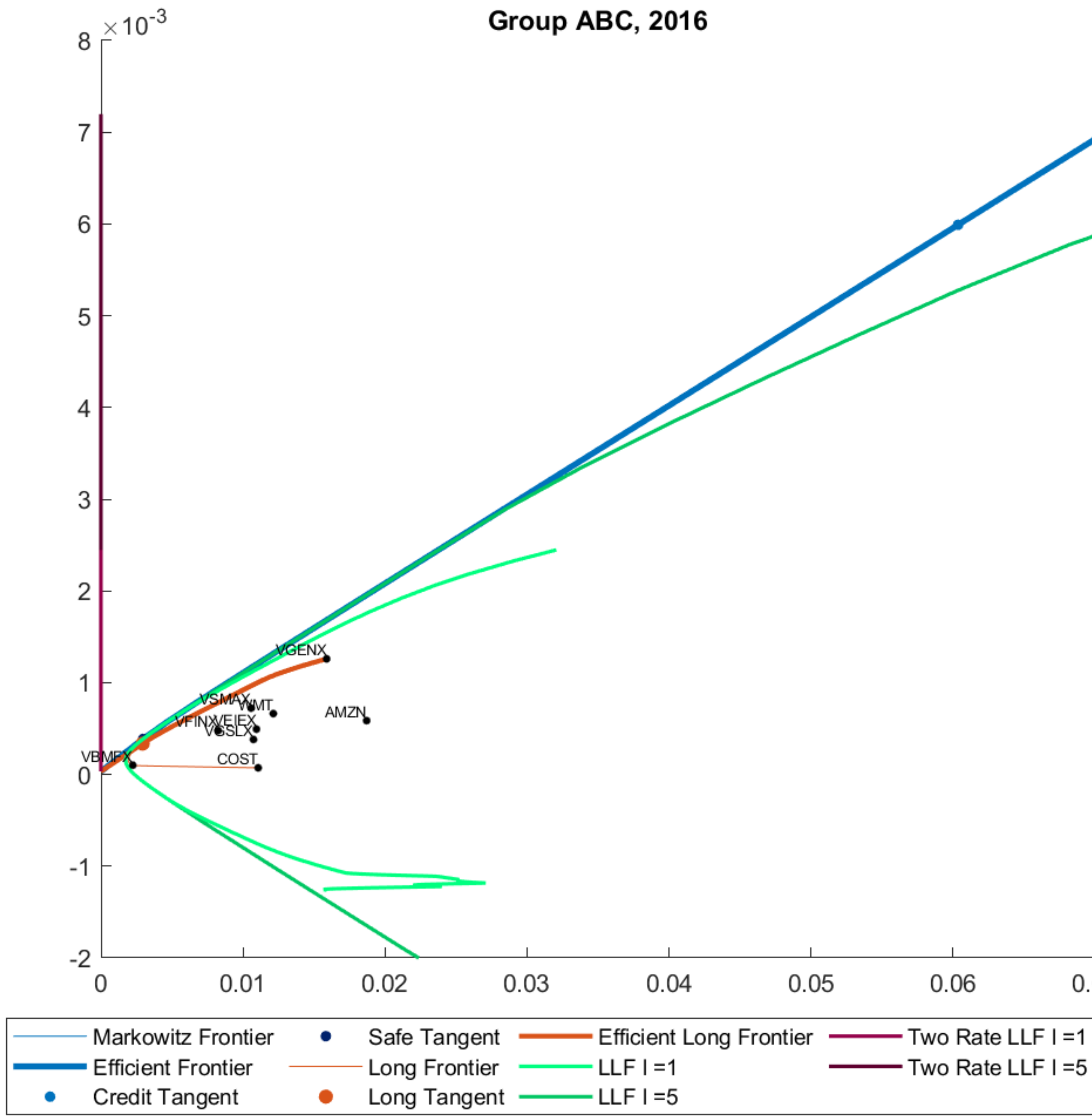

Group ABC, 2011
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
COST
WMT
VGSLX
VBMFX
VFINX
VGENX
VGMAX
AMZN
VEIEX
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

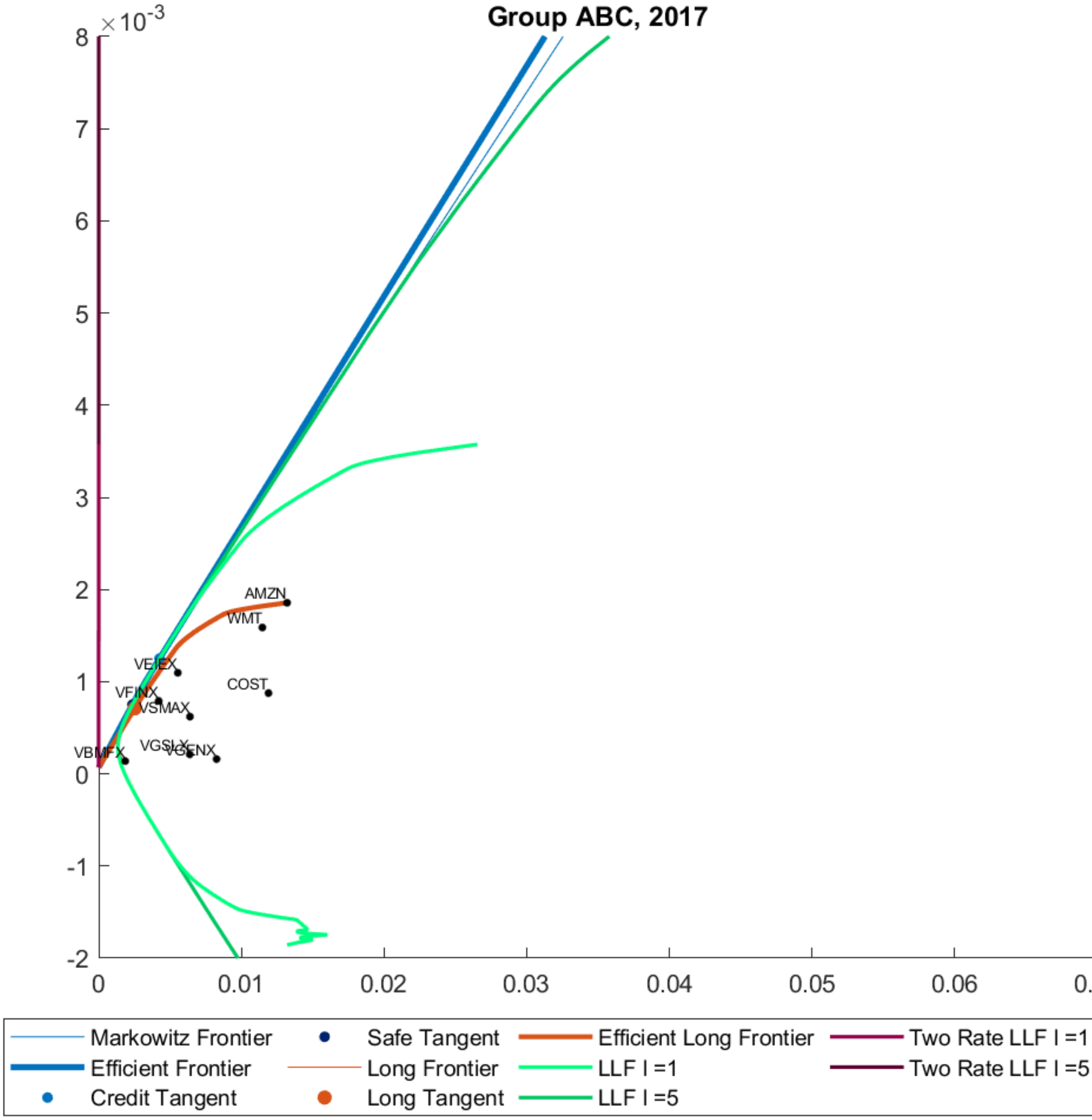

Group ABC, 2012
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
AMZN
COST
VFINX
VGENX
VBMFX
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5


Group ABC, 2013
×10^-3
AMZN
VSMAX
VFINX
COST
VGSLX
VEIEX
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

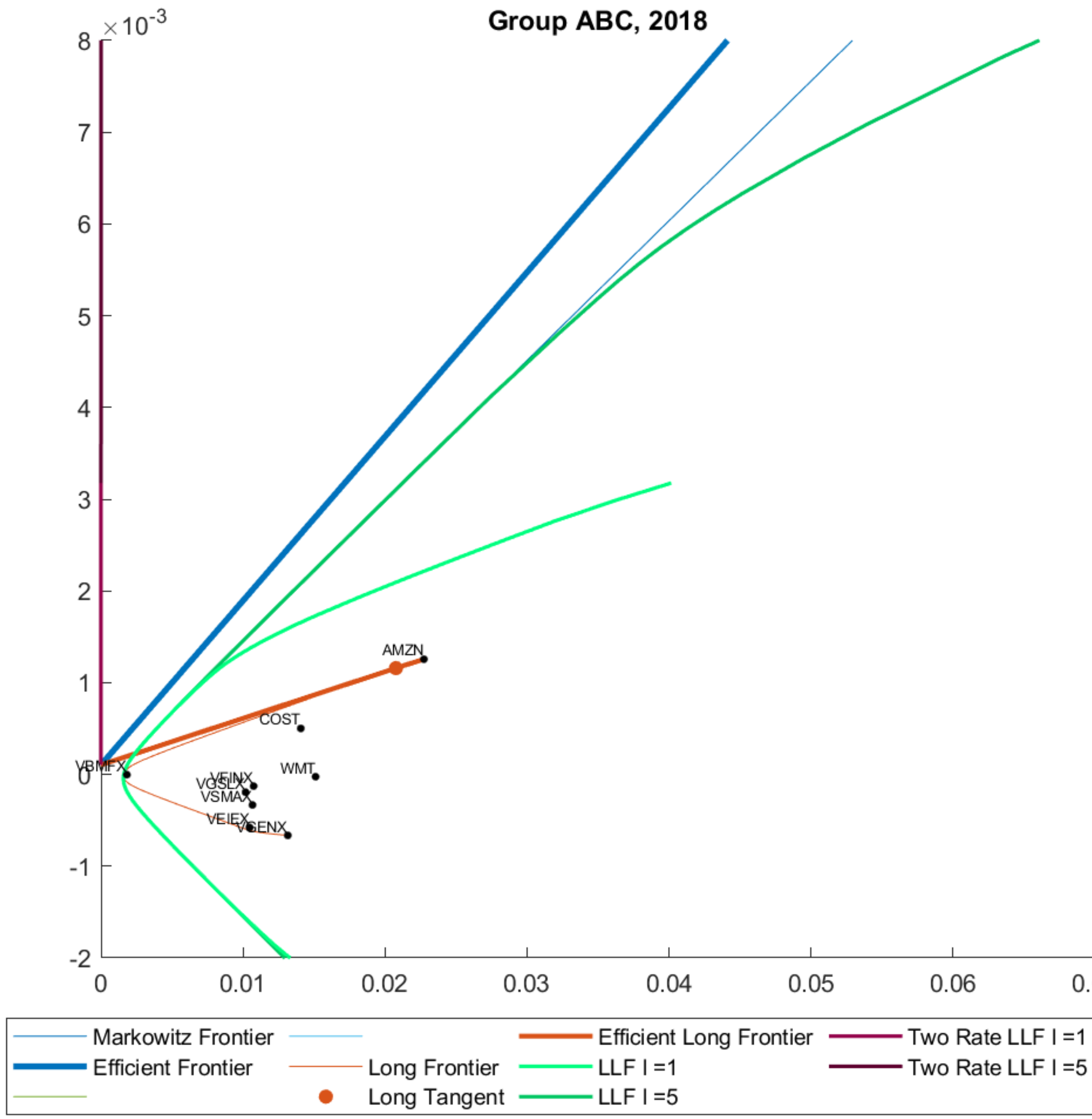

Group ABC, 2014
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
VGSLX
COST
VFINX
WMT
VSMAX
VBMFX
VEIEX
VGENX
AMZN
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

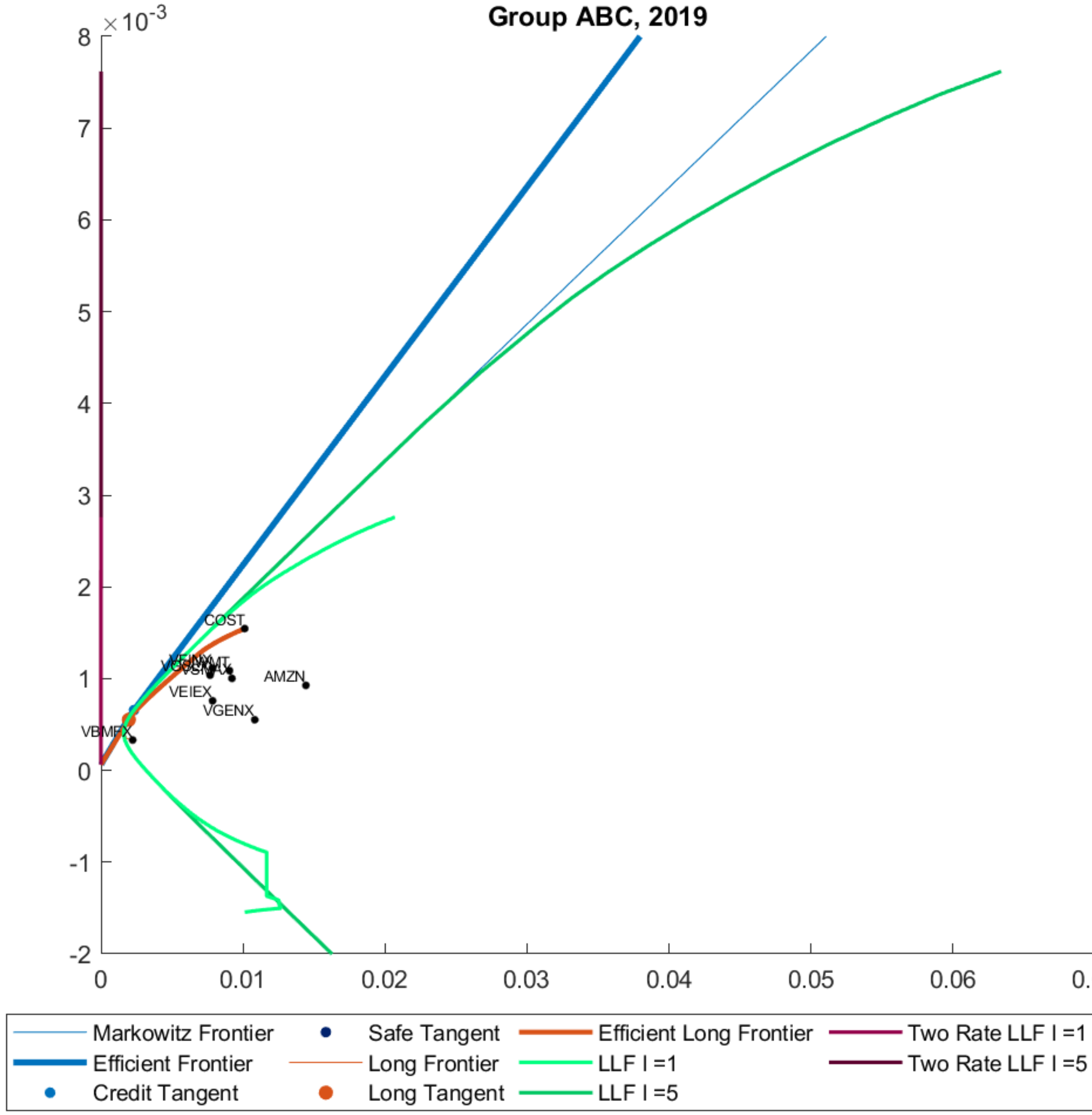

Group ABC, 2015
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
AMZN
COST
VGSLX
VFINX
VBMFX
VSMAX
VEIEX
VGENX
WMT
Markowitz Frontier
Efficient Frontier
Efficient Long Frontier
Long Frontier
Long Tangent
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

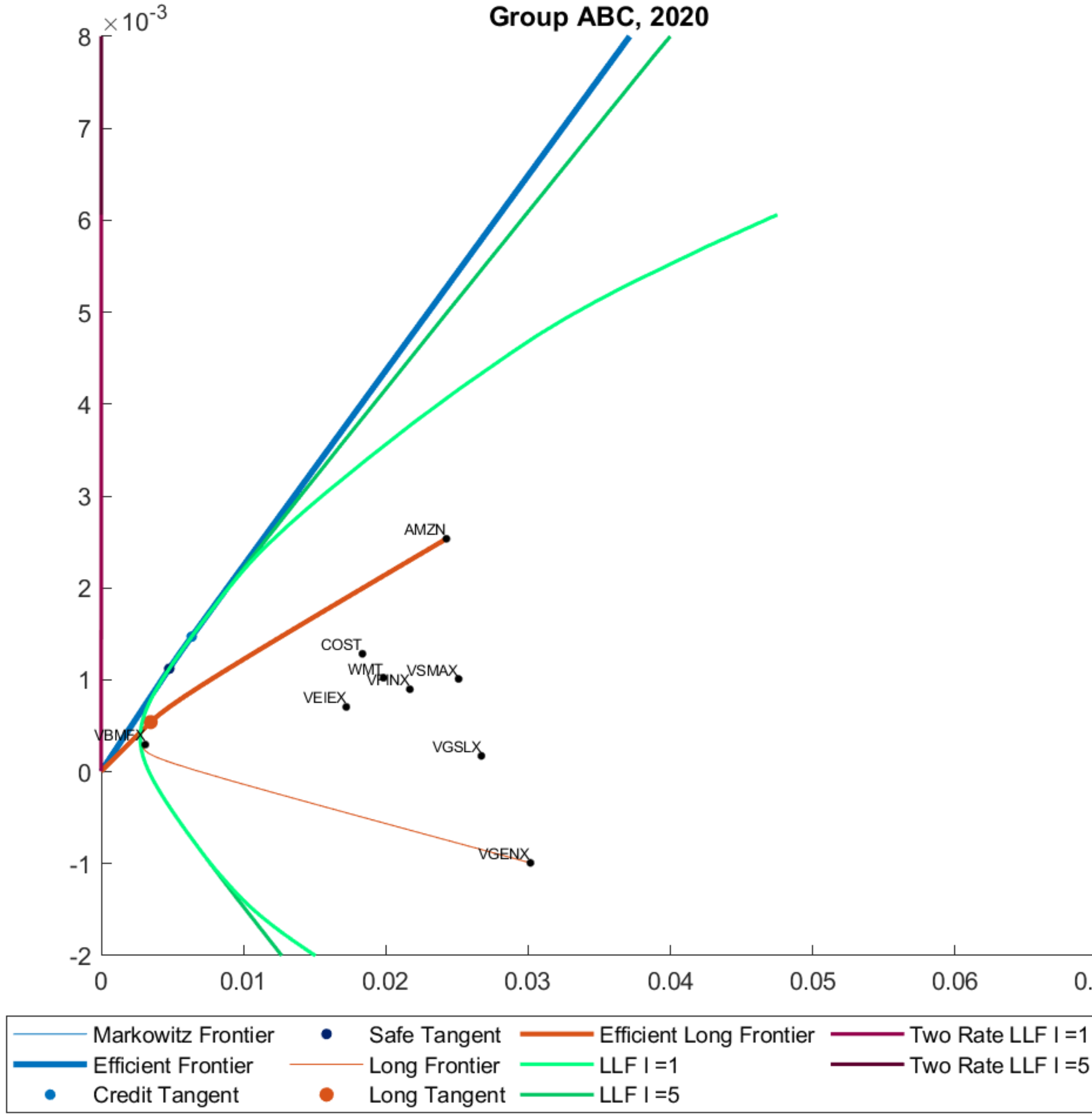

Group ABC, 2016
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
VGENX
VSMAX
WMT
AMZN
VFINX
VEIEX
VGSLX
VBMFX
COST
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5

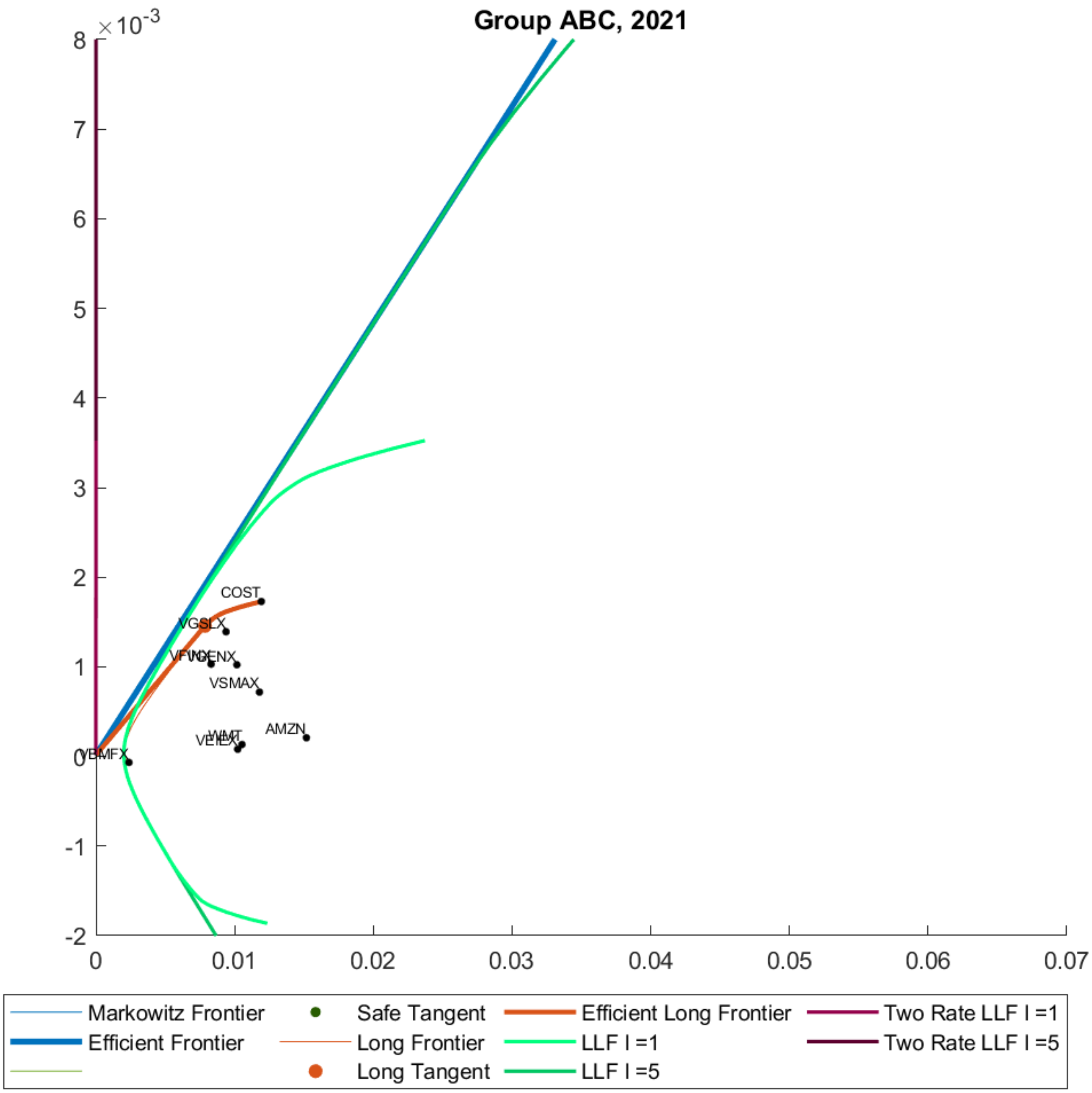


Group ABC, 2017
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
AMZN
WMT
VEIEX
COST
VFINX
VSMAX
VBMFX
VGSLX
VGENX
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5


Group ABC, 2018
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
AMZN
COST
WMT
VBMFX
VFINX
VGSLX
VSMAX
VEIEX
VGENX
Markowitz Frontier
Efficient Frontier
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5


Group ABC, 2019
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
COST
AMZN
VEIEX
VGENX
VBMFX
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5


Group ABC, 2020
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
AMZN
COST
WMT
VFINX
VSMAX
VEIEX
VBMFX
VGSLX
VGENX
Markowitz Frontier
Efficient Frontier
Credit Tangent
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5


Group ABC, 2021
×10^-3
8
7
6
5
4
3
2
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
COST
VGSLX
VSMAX
AMZN
VBMFX
Markowitz Frontier
Efficient Frontier
Safe Tangent
Long Frontier
Long Tangent
Efficient Long Frontier
LLF l =1
LLF l =5
Two Rate LLF l =1
Two Rate LLF l =5