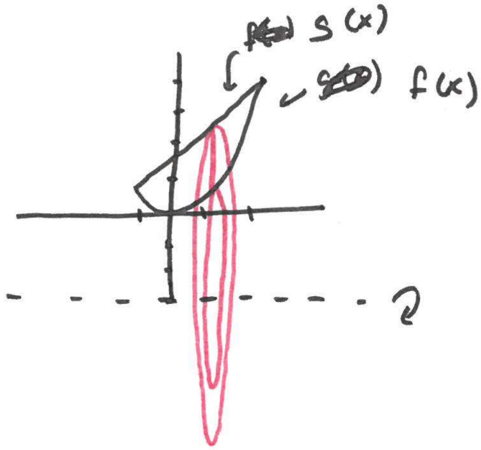
Study Session 1 - Spring 2016

1a) $x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$: points of intersection



$f(2) = 4$, $f(-1) = 1$

$f(0) = 0$, $g(0) = 2$, $g > f$

$\cancel{f}$ $\cancel{A(x) = \pi[(f(x) - g)^2 - g}$

$$A(x) = \pi\left[(g(x) - 3)^2 - (f(x) - 3)^2\right] = \pi\left[(x-1)^2 - (x^2 - 3)^2\right]$$

$$= \pi\left[x^2 - 2x + -x^4 + 6x^2 - 9\right]$$

$$V = \int_{-1}^{2} A(x)\,dx = \int_{-1}^{2} \pi\left[-x^4 + 7x^2 - 2x - 9\right] dx$$

1b) $L = \int_1^3 \sqrt{1 + [g(x)]^2}\,dx$

$g(x) = x^3 + \frac{1}{12x}$

$g'(x) = 3x^2 - \frac{1}{12x^2}$

$[g'(x)]^2 = (3x^2)^2 + \left(\frac{1}{12x^2}\right)^2 - 2\,\frac{3x^2}{12x^2}$

$= (3x^2)^2 + \left(\frac{1}{12x^2}\right)^2 - \frac{1}{2}$

$1 + [g'(x)]^2 = (3x^2)^2 + \left(\frac{1}{12x^2}\right)^2 + \frac{1}{2}$

$= \left(3x^2 + \frac{1}{12x^2}\right)^2$

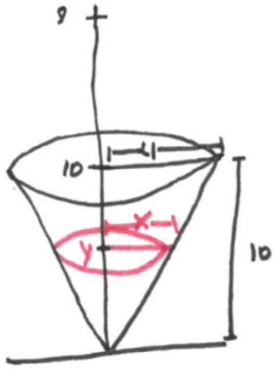
$= \int_1^3 3x^2 + \frac{1}{12x^2}\,dx$

$= \left[x^3 - \frac{1}{12x}\right]_1^3$

$= \left[27 - \frac{1}{36}\right] - \left[1 - \frac{1}{12}\right]$

$= 26 + \frac{2}{36} = 26 + \frac{1}{18}$

2a]



$y = \frac{10}{4} x$

$x = \frac{4}{10} y$

$A(y) = \pi r^2 = \pi x^2 = \pi \left(\frac{4}{10} y\right)^2$

$= \pi \frac{16}{100} y^2 = \frac{16}{100} \pi y^2$

$W = \int_0^{10} (8-y)\, 62.5\, A(y)\, dy = \int_0^{10} 62.5 \left(\frac{16}{100}\pi\right)(8-y) y^2\, dy$

2b]

Symmetric about $x=0$, so $\bar{x} = 0$

$\bar{y} = \frac{M_x}{A} = \frac{\frac{64}{10}}{\frac{16}{3}} = \frac{64}{10} \frac{3}{16} = \frac{12}{10} = \frac{6}{5}$

$M_x = \int_{-2}^{2} \frac{1}{2}\left[(x^2)^2\right] dx = \int_{-2}^{2} \frac{1}{2} x^4\, dx$

$= \left[\frac{1}{10} x^5\right]_{-2}^{2}$

$= \frac{1}{10}\left[2^5 - (-2)^5\right]$

$= \frac{64}{10}$

$A = \int_{-2}^{2} [x^2]\, dx = \left[\frac{1}{3} x^3\right]_{-2}^{2} = \frac{1}{3}\left[2^3 - (-2)^3\right]$

$= \frac{16}{3}$

3a] $f'(x) = 1 + \frac{1}{2\sqrt{x}} > 0$ for $x > 0$. So f is always increasing on its domain, and must therefore have an inverse.

Note $f(1) = 2$, so,

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

3b] $$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{\ln(5)} \ln\left(\tan^{-1}(x^3)\right)\right]$$

$$= \frac{1}{\ln(5)\tan^{-1}(x^3)} \frac{d}{dx}\left[\tan^{-1}(x^3)\right]$$

$$= \frac{1}{\ln(5)\tan^{-1}(x^3)} \frac{1}{1+(x^3)^2} \frac{d}{dx}\left[x^3\right]$$

$$= \frac{1}{\ln(5)\tan^{-1}(x^3)} \frac{3x^2}{1+x^6}$$

4a) $\displaystyle\int \frac{1}{(x+2)(x+1)}\,dx$

$$= \int \frac{-1}{x+2} + \frac{1}{x+1}\,dx$$

$$= -\ln|x+2| + \ln|x+1| + C$$

$$\frac{1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$A(x+1) + B(x+2) = 1$$

$x:\quad A + B = 0 \implies A = -B$

$1:\quad A + 2B = 1$

$-B + 2B = 1 \implies B = 1$

$A = -1$

4b) $\displaystyle\int_{-1}^{4} \frac{1}{(x+2)(x+1)}\,dx$

Continuity error at $x = -1$

$$= \lim_{a \to -1^-} \int_a^4 \frac{1}{(x+2)(x+1)}\,dx$$

$$= \lim_{a \to -1^-} \Big[\ln|x+1| - \ln|x+2|\Big]_a^4$$

$$= \lim_{a \to -1^-} \left[\ln\left|\frac{x+1}{x+2}\right|\right]_a^4$$

$$= \lim_{a \to -1^-} \left[\ln\left|\frac{5}{6}\right| - \underbrace{\ln\left|\frac{\overset{\to 0}{a+1}}{a+2}\right|}_{-\infty}\right]$$

$$= \infty$$

The integral diverges.

5a) $$\int_0^{\pi/4} t \sin(2t)\, dt$$

$u = t$ $\quad dv = \sin(2t)\,dt$

$du = dt$ $\quad v = -\frac{1}{2}\cos(2t)$

$$= \left[-\frac{1}{2} t \cos(2t)\right]_0^{\pi/4} + \int_0^{\pi/4} \frac{1}{2}\cos(2t)\, dt$$

$$= \left[-\frac{1}{2}\frac{\pi}{4}\cancel{\cos\left(\frac{\pi}{2}\right)}^{\to 0} + \frac{1}{2}(0)\cos(0)\right] + \left[\frac{1}{4}\sin(2t)\right]_0^{\pi/4}$$

$$= \left[\frac{1}{4}\sin(\pi/2) - \frac{1}{4}\sin(0)\right] = \frac{1}{4}$$

5b) $$\int_{-2}^{2\sqrt{3}-2} \frac{1}{t^2+4t+8}\, dt = \int_{-2}^{2\sqrt{3}-2} \frac{1}{(t^2+4t+4)+4}\, dt$$

$$= \int_{-2}^{2\sqrt{3}-2} \frac{1}{(t+2)^2+4}\, dt$$

$x = t+2$

$dx = dt$

$$= \int_0^{2\sqrt{3}} \frac{1}{x^2+4}\, dx = \left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^{2\sqrt{3}}$$

$$= \left[\frac{1}{2}\arctan(\sqrt{3})\right] - \cancel{\left[\frac{1}{2}\arctan(0)\right]}^{\to 0}$$

$$= \frac{1}{2}\frac{\pi}{3} = \frac{\pi}{6}$$

6a]

$$\lim_{x \to 0^+} \frac{\overbrace{\ln(\overset{\to 0^+}{x})}^{\to -\infty}}{\underbrace{\ln(\underbrace{\sin x}_{0^+})}_{-\infty}} \overset{LH}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{\cos x}{\sin x}} = \lim_{x \to 0^+} \frac{\sin x}{x \cos x}$$

$$= \left(\lim_{x \to 0^+} \frac{\sin x}{x}\right) \underbrace{\left(\lim_{x \to 0^+} \frac{1}{\underbrace{\cos x}_{1}}\right)}_{1} = \lim_{x \to 0^+} \frac{\overbrace{\sin x}^{\to 0}}{\underbrace{x}_{0}} \overset{LH}{=} \lim_{x \to 0^+} \frac{\overbrace{\cos(x)}^{\to 1}}{1} = 1$$

6b]

$$\lim_{n \to \infty} \left(1 - \frac{3}{2n}\right)^{2n} = e^{\lim_{n \to \infty} 2n \ln\left(1 - \frac{3}{2n}\right)} = e^{-3}$$

$$\lim_{n \to \infty} 2n \ln\left(1 - \frac{3}{2n}\right) = \lim_{n \to \infty} \frac{\overbrace{\ln\left(\overbrace{1 - \frac{3}{2n}}^{\to 1}\right)}^{\to 0}}{\underbrace{\frac{1}{2n}}_{0}} \overset{LH}{=} \lim_{n \to \infty} \frac{\frac{1}{1 - \frac{3}{2n}} \quad \frac{3}{2n^2}}{-\frac{1}{2n^2}}$$

$$= \lim_{n \to \infty} \frac{-3}{\underbrace{1 - \frac{3}{2n}}_{1}} = -3$$

7a) $$\sum_{n=2}^{\infty} \frac{2^{2n}}{(24)^n} = \sum_{n=2}^{\infty} \frac{4^n}{(24)^n} = \sum_{n=2}^{\infty} \left(\frac{4}{24}\right)^n = \sum_{n=2}^{\infty} \left(\frac{1}{6}\right)^n$$

$$= \frac{\left(\frac{1}{6}\right)^2}{1-\frac{1}{6}} = \frac{\left(\frac{1}{6}\right)^2}{\frac{5}{6}} = \frac{\frac{1}{6}}{5} = \frac{1}{30}$$

7b) $$\sum_{k=1}^{\infty} (-1)^k (k+5)^{-1/2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+5)^{1/2}}$$

$$b_k = \frac{1}{k^{1/2}}$$

$$\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| = \lim_{k\to\infty} \left| \frac{\frac{(-1)^k}{(k+5)^{1/2}}}{\frac{1}{k^{1/2}}} \right| = \lim_{k\to\infty} \frac{k^{1/2}}{(k+5)^{1/2}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k+5}} \cdot \frac{\frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{1}{\sqrt{1+\underbrace{\frac{5}{\sqrt{k}}}_{\to 0}}} = 1$$

$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ diverges by p-series test since $p = 1/2 \leq 1$. So, by generalized limit comparison test, $\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+5)^{1/2}}$ does <u>not</u> converge absolutely.

$$\lim_{k\to\infty} \frac{(-1)^k}{\underbrace{(k+5)^{1/2}}_{\to\infty}} = 0$$

And, $(k+5) < ((k+1)+5)$

$$(k+5)^{1/2} < ((k+1)+5)^{1/2}$$

$$\frac{1}{(k+5)^{1/2}} > \frac{1}{((k+1)+5)^{1/2}}$$ Decreasing

By the Alternating Series Test, <u>converges conditionally</u>

8a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

$\frac{1}{1+2x} = \sum_{n=0}^{\infty} (-2x)^n$ for $|-2x| = |2x| < 1$

$|x| < \frac{1}{2}$

$= \sum_{n=0}^{\infty} (-1)^n 2^n x^n$

$f(x) = \frac{x^2}{1+2x} = x^2 \sum_{n=0}^{\infty} (-1)^n 2^n x^n$

$= \sum_{n=0}^{\infty} (-1)^n 2^n x^{n+2}$ for $|x| < \frac{1}{2}$, so $R_1 = \frac{1}{2}$

$f'(x) = \frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n 2^n x^{n+2}\right]$

$= \cancel{\frac{d}{dx}} \sum_{n=0}^{\infty} \frac{d}{dx}\left[(-1)^n 2^n x^{n+2}\right]$

$= \sum_{n=0}^{\infty} (-1)^n 2^n (n+2) x^{n+1}$ $R_2 = R_1 = \frac{1}{2}$

8b) $f(x) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$P_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$

$P(2/3) = \frac{2}{3} - \frac{(\frac{2}{3})^3}{6}$

$= \frac{2}{3} - \frac{8}{27 \cdot 6}$

$= \frac{2}{3} - \frac{8}{162} = \frac{50}{81}$

$\sin(x) = P_3(x) + \frac{f^{(4)}(x^*)}{4!} x^4$

$= P_3\left(\frac{2}{3}\right) + \frac{f^{(4)}(x^*)}{4!} \left(\frac{2}{3}\right)^4$

$x^* \in \left[0, \frac{2}{3}\right]$

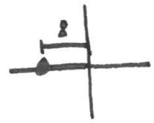
$\left|f^{(4)}(x^*)\right| < 1$

Error $< \frac{1}{4!}\left(\frac{2}{3}\right)^4$

$= \frac{2}{243}$

9a) $z^3 = -8$

$= 8e^{i\pi}$

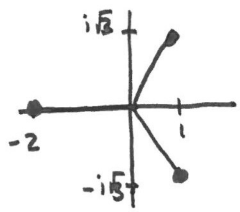


$$z_k = 8^{1/3} e^{i\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right)} = 2e^{i\left(\frac{(2k+1)\pi}{3}\right)}$$

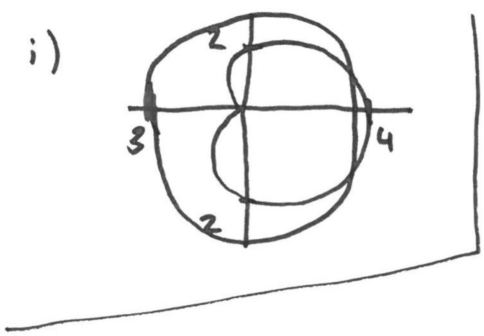
$$z_0 = 2e^{i\frac{\pi}{3}} = 2\cos\left(\frac{\pi}{3}\right) + 2i\sin\left(\frac{\pi}{3}\right) = 1 + i\sqrt{3}$$

$$z_1 = 2e^{i(\pi)} = -2$$

$$z_2 = 2e^{i\left(\frac{5\pi}{3}\right)} = 2\cos\left(\frac{5\pi}{3}\right) + 2i\sin\left(\frac{5\pi}{3}\right) = 1 - i\sqrt{3}$$



9b) i)



ii) $3 = 2(1 + \cos\theta)$

$3 = 2 + 2\cos\theta$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta \implies \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$$A = \int_{\pi/3}^{5\pi/3} \frac{1}{2}\left[3^2 - (2(1+\cos\theta))^2\right] d\theta$$