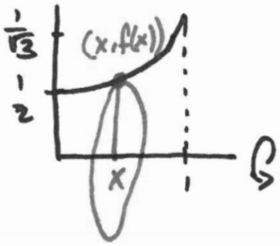
Study Session 2 - Spring 2019

1a]



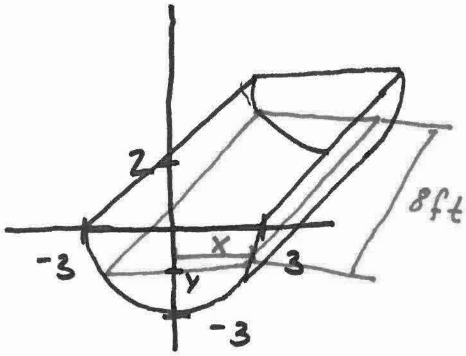
$$A(x) = \pi\left[f(x)\right]^2$$
$$= \pi\left(\frac{1}{\sqrt{4-x}}\right)^2 = \frac{\pi}{4-x}$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \frac{\pi}{4-x}\,dx \qquad u = 4-x \quad du = -dx$$
$$= -\int_4^3 \frac{\pi}{u}\,du = -\left[\ln|u|\right]_4^3$$
$$= -(\ln(3) - \ln(4))$$
$$= \ln(4) - \ln(3) = \ln\left(\frac{4}{3}\right)$$

1b] $$L = \int_0^4 \sqrt{1 + [g'(x)]^2}\,dx \qquad g(x) = \frac{1}{3}x^{3/2} - 1$$
$$g'(x) = \frac{1}{2}x^{1/2}$$
$$= \int_0^4 \sqrt{1 + \frac{1}{4}x}\,dx \qquad u = 1 + \frac{1}{4}x \quad [g'(x)]^2 = \frac{1}{4}x$$
$$du = \frac{1}{4}dx$$
$$dx = 4du$$
$$= 4\int_1^2 \sqrt{u}\,du$$

$$= 4\left[\frac{2}{3}u^{3/2}\right]_1^2 = 4\left[\frac{2}{3}(2)^{3/2} - \frac{2}{3}\right] = \frac{8}{3}(2)^{3/2} - \frac{8}{3}$$

2a)



$A(y) = 8(2x) = 16x$

$= 16\sqrt{9-y^2}$

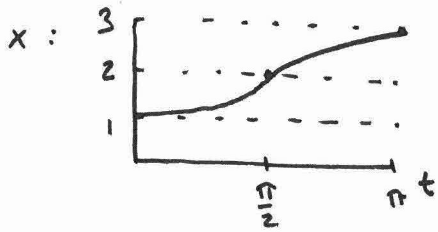
$x^2 + y^2 = 9$

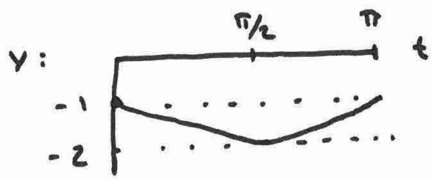
$x^2 = 9 - y^2$

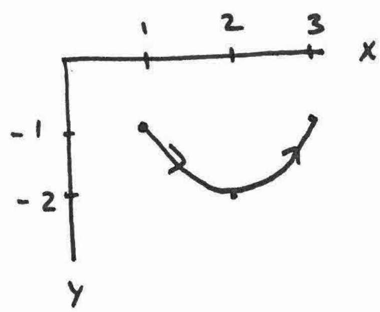
$x = \sqrt{9-y^2}$

$$W = \int_{-3}^{0} (2-y)\, 62.5\, A(y)\, dy = \int_{-3}^{0} 62.5\, (2-y)\left(16 - \sqrt{9-y^2}\right) dy$$

2b)



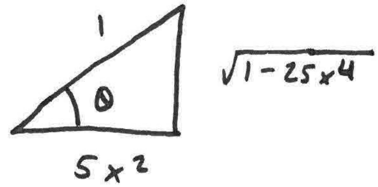




3a] $\cot(\cos^{-1}(5x^2))$ Let $\theta = \cos^{-1}(5x^2)$

$= \cot(\theta)$ $\cos(\theta) = 5x^2 = \frac{5x^2}{1} = \frac{A}{H}$

$= \frac{5x^2}{\sqrt{1-25x^4}}$



3b] $f'(x) = -1 - \frac{1}{\sqrt{16-x}} < 0$ for $x \in (-\infty, 16)$

$I = \cancel{x\in} (-\infty, 16]$ $f(16) = -16$

Note that $f(0) = 4$, so

$$(f^{-1})'(4) = \frac{1}{f'(0)} = \frac{1}{-1-\frac{1}{\sqrt{16-0}}} = \frac{1}{-1-\frac{1}{4}} = \frac{1}{-\frac{5}{4}} = -\frac{4}{5}$$

4a] $\displaystyle\int \frac{\sin t}{\sqrt{4-16\cos^2 t}}\,dt \quad = \int \frac{\sin t}{\sqrt{4-(4\cos t)^2}}\,dt$

$u = 4\cos t$

$du = -4\sin t\; dt$

$dt = \dfrac{du}{-4\sin t}$

$$= -\frac{1}{4}\int \frac{1}{\sqrt{4-u^2}}\,du$$

$$= -\frac{1}{4}\int \frac{1}{\sqrt{2^2-u^2}}\,du$$

$$= -\frac{1}{4}\arcsin\left(\frac{1}{2}u\right) + c$$

$$= -\frac{1}{4}\arcsin\left(\frac{1}{2}4\cos t\right) + c$$

$$= -\frac{1}{4}\arcsin\left(2\cos t\right) + c$$

4b] $\displaystyle\int \frac{x^3+4x+3}{x^2+4}\,dx \quad = \int \frac{x(x^2+4)\;+3}{x^2+4}\,dx$

$$= \int x + \frac{3}{x^2+4}\,dx \quad = \int x\,dx + \int \frac{3}{x^2+4}\,dx$$

$$= \frac{1}{2}x^2 + \frac{3}{2}\arctan\left(\frac{x}{2}\right) + c$$

5a] $\int_1^{\infty} \frac{1}{x^3+2\sqrt{x}}\, dx$

for $x \in [1, \infty)$

$x^3 < x^3 + 2\sqrt{x}$

$\frac{1}{x^3} > \frac{1}{x^3+2\sqrt{x}}$

$\int_1^{\infty} \frac{1}{x^3}$ converges by p-test, so by comparison $\int_1^{\infty} \frac{1}{x^3+2\sqrt{x}}$

also converges.

5b] $\int_{-2}^{7} \frac{5}{\sqrt{t+2}}\, dt$

Continuity error at $t=-2$

$\int \frac{5}{\sqrt{t+2}}\, dt$ $\quad x = t+2$, $dx = dt$

$= \int \frac{5}{\sqrt{x}}\, dx$

$= 10\sqrt{x} + c = 10\sqrt{t+2} + c$

$= \lim_{a \to -2^+} \int_a^7 \frac{5}{\sqrt{t+2}}\, dt$

$= \lim_{a \to -2^+} \left[10\sqrt{t+2} \right]_a^7$

$= \lim_{a \to -2^+} \left[10\sqrt{9} - \underbrace{10\sqrt{a+2}}_{\to 0} \right]$

$= 30$

The integral converges.

6a) $K_T = \max_{x \in [1,3]} |f^{(2)}(x)|$

$f(x) = 6\ln(x)$

$f'(x) = \frac{6}{x}$

$f''(x) = -\frac{6}{x^2}$

$= \max_{x \in [1,3]} \frac{6}{x^2}$

$= 6$

$$E_n^T \le \frac{6(2)^3}{12n^2} \underset{\text{want}}{\le} \frac{1}{25}$$

$$\frac{(25)\,48}{12} \le n^2$$

$$100 \le n^2$$

$$10 \le n$$

6b) $\lim_{x\to\infty} \left(1+\frac{2}{x^2}\right)^{3x^2}$ $\quad y = x^2$, $\lim_{x\to\infty} y = \infty$

$$= \lim_{y\to\infty}\left(1+\frac{2}{y}\right)^{3y} = e^{\lim_{y\to\infty} 3y\ln\left(1+\frac{2}{y}\right)} = e^6$$

$$\lim_{y\to\infty} 3y\ln\left(1+\frac{2}{y}\right) = \lim_{y\to\infty} \frac{\overbrace{\ln\left(1+\frac{2}{y}\right)}^{\to 0}}{\underbrace{\frac{1}{3y}}_{\to 0}} \overset{LH}{=} \lim_{y\to\infty} \frac{\frac{1}{1+\frac{2}{y}}\cdot\frac{-2}{y^2}}{-\frac{1}{3y^2}}$$

$$= \lim_{y\to\infty} \frac{6}{\underbrace{1+\frac{2}{y}}_{\to 1}} = 6$$

6c) $$\lim_{x\to 0} \frac{\overbrace{\tan x - x}^{\to 0}}{\underbrace{\sin x - x}_{\to 0}} \overset{LH}{=} \lim_{x\to 0} \frac{\overbrace{\sec^2(x) - 1}^{\to 0}}{\underbrace{\cos(x) - 1}_{\to 0}} \overset{LH}{=} \lim_{x\to 0} \frac{2\sec(x)(\sec(x)\tan x)}{\sin x}$$

$$= \lim_{x\to\infty} \frac{2\sec^2(x)\frac{\cancel{\sin(x)}}{\cos(x)}}{\cancel{\sin(x)}} = \lim_{x\to\infty} \underbrace{2\sec^3(x)}_{\to 1} = 2$$

7a] $$\lim_{n\to\infty} \sqrt{n^2+2n} - \sqrt{n^2-2n} \left(\frac{\sqrt{n^2+2n} + \sqrt{n^2-2n}}{\sqrt{n^2+2n} + \sqrt{n^2-2n}} \right)$$

$$= \lim_{n\to\infty} \frac{n^2+2n - n^2 + 2n}{\sqrt{n^2+2n} + \sqrt{n^2+2n}} = \lim_{n\to\infty} \frac{4n}{\sqrt{n^2+2n} + \sqrt{n^2-2n}} \; \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{4}{\underbrace{\sqrt{1+\frac{2}{n}}}_{1} + \underbrace{\sqrt{1-\frac{2}{n}}}_{1}} = \frac{4}{2} = 2$$

7b] $$\sum_{n=2}^{\infty} \frac{2^n+3^n}{6^n} = \sum_{n=2}^{\infty} \frac{2^n}{6^n} + \sum_{n=2}^{\infty} \frac{3^n}{6^n} = \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{\left(\frac{1}{3}\right)^2}{1-\frac{1}{3}} + \frac{\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} = \frac{\frac{1}{9}}{\frac{2}{3}} + \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$$

8a] $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}+2}$ $\quad \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt{n}+2}_{\infty}} = 0$

$$n < n+1$$
$$\sqrt{n} < \sqrt{n+1}$$
$$\sqrt{n}+2 < \sqrt{n+1}+2$$

$\frac{1}{\sqrt{n}+2} > \frac{1}{\sqrt{n+1}+2}$, so decreasing, By AST, the series converges

want

$$E_j \le \frac{1}{\sqrt{j+1}+2} \le \frac{1}{5}$$

$$5 \le \sqrt{j+1}+2$$
$$3 \le \sqrt{j+1}$$
$$9 \le j+1$$
$$\boxed{8} \le j$$

8b] $\sum_{n=0}^{\infty} \frac{1}{(n+1)4^n} x^{2n}$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+2)4^{n+1}} x^{2(n+1)}}{\frac{1}{(n+1)4^n} x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(n+2)4^{n+1}} \frac{(n+1)4^n}{x^{2n}} \right|$$

$$= \lim_{n\to\infty} \frac{x^2}{4} \frac{n+1}{n+2} = \frac{x^2}{4} \lim_{n\to\infty} \frac{n+1}{n+2} \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{x^2}{4} \lim \frac{\overbrace{1+\frac{1}{n}}^{\to 1}}{\underbrace{1+\frac{2}{n}}_{\to 1}}$$

$= \frac{x^2}{4} < 1 \implies x^2 < 4 \implies |x| < 2$ converges by Generalized Ratio Test

$x=2$: $\sum_{n=0}^{\infty} \frac{1}{(n+1)4^n} (2)^{2n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)4^n} 4^n = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$

Diverges by harmonic series

$x=-2$: $\sum_{n=0}^{\infty} \frac{1}{(n+1)4^n} (-2)^{2n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)4^n} 4^n$ diverges by above

$$I = (-2, 2)$$

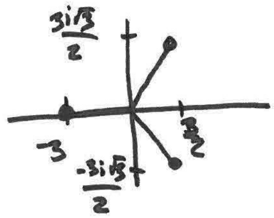
$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{1}{(n+1)4^n} x^{2n}\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[\frac{1}{(n+1)4^n} x^{2n}\right] = \sum_{n=0}^{\infty} \frac{2n}{(n+1)4^n} x^{2n-1}$$

9a]

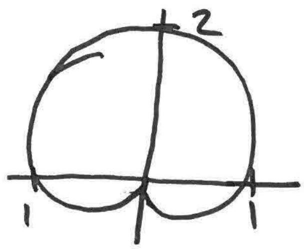
$$-27 = 27e^{i\pi}$$

$$z_k = (27)^{1/3}\, e^{i\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right)} = 3e^{i\left(\frac{(2k+1)\pi}{3}\right)}$$

$$z_0 = 3e^{i\pi/3}, \quad z_1 = 3e^{i\pi} = -3, \quad z_2 = 3e^{i\frac{5\pi}{3}}$$



9b]



$$A = \int_0^{2\pi} \frac{1}{2}\left[1 + \sin\theta\right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \sin^2\theta + 2\sin\theta + 1 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2\theta) + 2\sin\theta + 1 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 2\sin\theta - \frac{1}{2}\cos(2\theta) + \frac{3}{2} \, d\theta$$

$$= \frac{1}{2}\left[-2\cos\theta - \frac{1}{4}\sin(2\theta) + \frac{3}{2}\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[-2 + 3\pi + 2\right] = \frac{3\pi}{2}$$