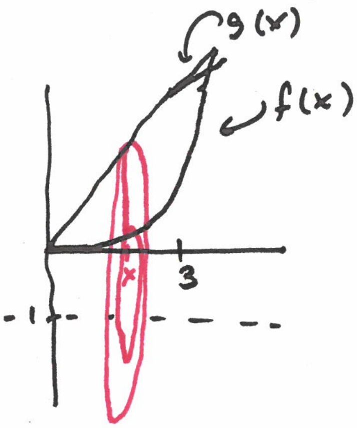
Study Session 3 - Spring 2022

1a] $f(x) = x^2 \qquad g(x) = 3x$

$x^2 = 3x \implies x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, 3$



$\cancel{f(0)=0} \qquad f(1) = 1 < g(1) = 3$

$$V = \int_a^b A(x)\,dx$$

$$= \int_0^3 A(x)\,dx$$

$$A(x) = \pi(R^2 - r^2) = \pi\left((g(x)+1)^2 - (f(x)+1)^2\right)$$

$$= \pi\left((3x+1)^2 - (x^2+1)^2\right)$$

$$V = \int_0^3 \pi\left((3x+1)^2 - (x^2+1)^2\right) dx$$

1b] $f(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

$$f'(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

$$[f'(x)]^2 = \left(\frac{1}{2}e^x\right)^2 + \left(\frac{1}{2}e^{-x}\right)^2 - 2\left(\frac{1}{2}e^x\right)\left(\frac{1}{2}e^{-x}\right)$$

$$= \left(\frac{1}{2}e^x\right)^2 + \left(\frac{1}{2}e^{-x}\right)^2 - \frac{1}{2}$$

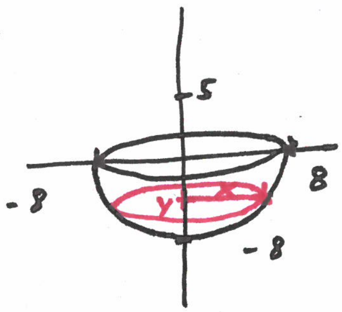
$$1 + [f'(x)]^2 = \left(\frac{1}{2}e^x\right)^2 + \left(\frac{1}{2}e^{-x}\right)^2 + \frac{1}{2} = \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^2$$

$$L = \int_0^1 \frac{1}{2}e^x + \frac{1}{2}e^{-x}\,dx = \left[\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right]_0^1$$

$$= \left[\frac{1}{2}e - \frac{1}{2}e^{-1}\right] - \left[\frac{1}{2}(1) - \frac{1}{2}(1)\right]$$

$$= \frac{1}{2}e - \frac{1}{2}e^{-1}$$

2a]



$$W = \int_a^b \rho\,(l-y)\,A(y)\,dy$$

$$= \int_{-6}^{0} 62.5\,(5-y)\,A(y)\,dy$$

$$= \int_{-6}^{0} 62.5\,(5-y)\,\pi(64-y^2)\,dy$$

$x^2+y^2=64$

$x^2=64-y^2$

$A(y) = \pi r^2$

$= \pi x^2$

$= \pi(64-y^2)$

2b] $f(x) = 11-x^2$ $\quad g(x) = x^2+3$ $\quad 11-x^2 = x^2+3 \implies 0 = 2x^2-8$

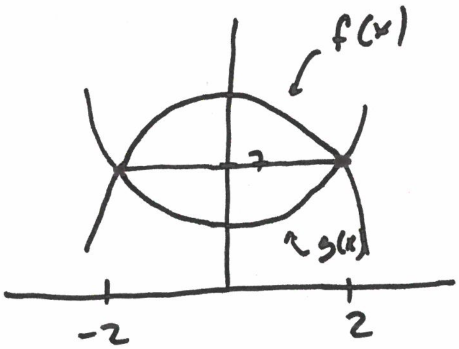
$0 = x^2-4$

$= (x+2)(x-2)$

$\cancel{x\ 2 0+}\ x = -2, 2$

$f(-2) = 7$

$f(2) = 7$



$f(0) = 11 > g(0) = 3$

Symmetric over $x=0$, so $\bar{x} = 0$

$\cancel{\bar{y} = \frac{M_x}{A}}$ Symmetric over $y=7$, so $\bar{y} = 7$

$(0, 7)$

$$\cancel{M_x = \int_{-2}^{2} \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right] dx}$$

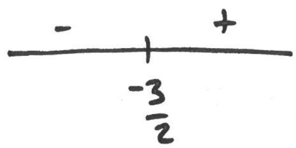
$$\cancel{= \int_{-2}^{2} \frac{1}{2}\left[(11-x^2)^2 - (x^2+3)^2\right] dx}$$

$$\cancel{= \int_{-2}^{2} \frac{1}{2}\left[(x^4-22x^2+121) - (x^4+6x^2+9)\right] dx}$$

$$\cancel{= \int_{-2}^{2} \frac{1}{2}\left[-16x^2 +\right.}$$

3a] $f'(x) = 2x+3$

$2x+3=0 \Rightarrow x = -\frac{3}{2}$



$f'(x) > 0$ for $x > -\frac{3}{2}$

so f is increasing on $(-\frac{3}{2}, \infty)$

So, f has an inverse on $[-3/2, \infty)$

3b] $g(x) = x^3 + x + 2$

$g'(x) = 3x^2 + 1$

$3x^2 \geq 0$, $1 > 0$ for all x, so $g'(x) > 0$ for all x

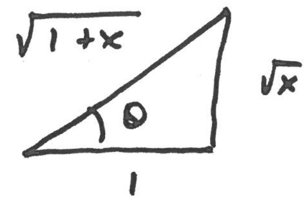
So g is always increasing, so it must have an inverse.

$g(2) = 12$, so $(g^{-1})'(12) = \frac{1}{g'(2)} = \frac{1}{3(2)^2+1} = \frac{1}{13}$

4a] $\sec(\tan^{-1}(\sqrt{x}))$

Let $\theta = \tan^{-1}(\sqrt{x})$

$\tan(\theta) = \sqrt{x} = \frac{\sqrt{x}}{1} = \frac{O}{A}$



$= \sec(\theta) = \frac{H}{A}$

$= \sqrt{1+x}$

4b] $\lim\limits_{x\to\infty}\left(1 - \frac{4}{x}\right)^x = e^{\lim\limits_{x\to\infty} x\ln\left(1-\frac{4}{x}\right)} = e^{-4}$

$$\lim_{x\to\infty} x\ln\left(1-\frac{4}{x}\right) = \lim_{x\to\infty}\frac{\overbrace{\ln(\overbrace{1-\tfrac{4}{x}}^{1})}^{\to 0}}{\underbrace{\frac{1}{x}}_{0}} \overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{1-\frac{4}{x}}\cdot\frac{4}{x^2}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{-4}{\underbrace{1-\cancel{4x}\,\frac{4}{x}}_{1}} = -4$$

5a] $\int_1^3 x^2 \ln(x)\,dx$

$u = \ln(x) \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{3} x^3 dx$

$$= \left[\frac{1}{3} x^3 \ln(x)\right]_1^3 - \int_1^3 \frac{1}{3} x^3 \cdot \frac{1}{x}\,dx$$

$$= \left[9 \ln(3) - 0\right] - \int_1^3 \frac{1}{3} x^2\,dx$$

$$= 9 \ln(3) - \left[\frac{1}{9} x^3\right]_1^3 = 9 \ln(3) - 3 + \frac{1}{9}$$

$$= 9 \ln(3) - \frac{26}{9}$$

5b] $\int_0^2 \sqrt{4-t^2}\,dt$ See Spring 2017 $\boxed{5a}$

$= \pi$

6a) $\displaystyle\int \frac{2x+6}{x(x+2)}\,dx$

$$\frac{2x+6}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

$$= \int \frac{3}{x} - \frac{1}{x+2}\,dx$$

$$2x+6 = A(x+2) + Bx$$

$$x = 0 : \quad 6 = 2A \implies A = 3$$

$$x = -2 : \quad 2 = -2B \implies B = -1$$

$$= 3\ln|x| - \ln|x+2| + C$$

6b) $\displaystyle\int_0^\infty 3^{-x}\,dx = \lim_{b\to\infty} \int_0^b 3^{-x}\,dx = \lim_{b\to\infty}\left[\frac{-1}{\ln(3)}\,3^{-x}\right]_0^\infty$

$$= \lim_{b\to\infty}\left[\underbrace{\frac{-1}{3^{b}\ln(3)}}_{\to 0} + \frac{1}{\ln(3)}\right]$$

$$= \frac{1}{\ln(3)}$$

~~Continui~~

$$\int 3^{-x}\,dx = \int e^{-x\ln(3)}\,dx$$

$$u = -\ln(3)x$$

$$du = -\ln(3)\,dx$$

$$dx = \frac{1}{-\ln(3)}\,du$$

$$= \int \frac{1}{-\ln(3)}\,e^u\,du = -\frac{1}{\ln(3)}\,e^u + C$$

$$= \frac{-1}{\ln(3)}\,e^{-\ln(3)x} + C = \frac{-1}{\ln(3)}\,3^{-x} + C$$

7a] $\lim_{n\to\infty}\left[\left(\sqrt{n+1}-\sqrt{n}\right)\sqrt{n+3}\right] \quad \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{\left((n+1)-n\right)\sqrt{n+3}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n+3}}{\sqrt{n+1}+\sqrt{n}} \; \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{3}{n}}\to 0}{\sqrt{1+\underset{\downarrow 0}{\frac{1}{n}}}+1} = \frac{1}{2} = \cancel{\tfrac{1}{2}} \quad \boxed{\frac{1}{2}}$$

7b] $\sum_{n=1}^{\infty} \frac{(-2)^n}{n\,3^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n\, 2^n}{n\; 3^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}\,2^{n+1}}{(n+1)\;3^{n+2}}}{\frac{(-1)^n\;2^n}{n\;3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,2^{n+1}}{(n+1)\,3^{n+2}}\;\frac{n\;3^{n+1}}{(-1)^n\,2^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\;\;2}{\cancel{(n+2)}\;\;3}\;\frac{n}{n+1}\right| = \frac{2}{3}\lim_{n\to\infty}\frac{n}{n+1}\;\frac{\frac{1}{n}}{\frac{1}{n}} = \frac{2}{3}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}$$

$= \frac{2}{3} < 1$ By Generalized Ratio Test, the series

<u>converges absolutely</u>

8a) $f(x) = \sum_{n=0}^{\infty} \frac{n^2}{5^n} x^{2n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{5^{n+1}} x^{2(n+1)}}{\frac{n^2}{5^n} x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 x^{2n+2}}{5^{n+1}} \frac{5^n}{n^2 x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{5} \frac{n^2}{(n+1)^2} \right| = \frac{x^2}{5} \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \frac{x^2}{5} \lim_{n\to\infty} \frac{1}{1 + \underbrace{\frac{2}{n} + \frac{1}{n^2}}_{\to 0}} = \frac{x^2}{5} < 1 \implies x^2 < 5$$

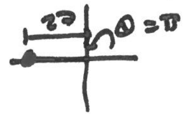
$$|x| < \sqrt{5}$$

By Generalized Ratio Test, the radius of convergence is $R = \sqrt{5}$

8b) $$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{n^2}{5^n} x^{2n}\right] = \sum_{n=0}^{\infty} \frac{n^2}{5^n} \frac{d}{dx}\left[x^{2n}\right]$$

$$= \sum_{n=1}^{\infty} \frac{n^2}{5^n} (2n) x^{2n-1} = \sum_{n=1}^{\infty} \frac{2n^3}{5^n} x^{2n-1}$$

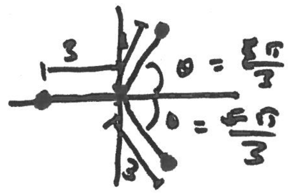
9a) $z = -27$



$= 27\, e^{i\pi}$

$z_k^3 = z, \quad z_k = (27)^{1/3}\, e^{i\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right)} = 3\, e^{i\left(\frac{\pi(2k+1)}{3}\right)}$

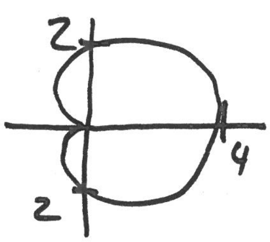
$z_0 = 3e^{i\frac{\pi}{3}}$



$z_1 = 3e^{i\pi}$

$z_2 = 3e^{i\frac{5\pi}{3}}$

9b)



$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2 d\theta$

$= \int_0^\pi \frac{1}{2}\left(2(1+\cos\theta)\right)^2 d\theta$

$= \int_0^\pi \frac{1}{2}(4)(1+\cos\theta)^2 d\theta$

$= 2\int_0^\pi \cos^2\theta + 2\cos\theta + 1\, d\theta$

$= 2\int_0^\pi \frac{1}{2} + \frac{1}{2}\cos(2\theta) + 2\cos\theta + 1\, d\theta$

$= 2\int_0^\pi \frac{1}{2}\cos(2\theta) + 2\cos\theta + \frac{3}{2}\, d\theta$

$= \int_0^\pi \cos(2\theta) + 4\cos\theta + 3\, d\theta$

$= \left[\frac{1}{2}\sin(2\theta) + 4\sin(\theta) + 3\theta\right]_0^\pi$

$= \left[0 + 0 + 3\pi\right] - \left[0 + 0 + 0\right] = \boxed{3\pi}$