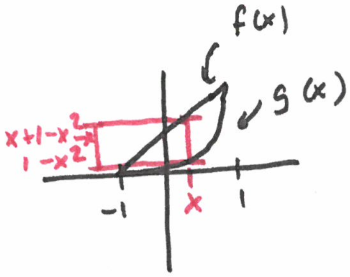
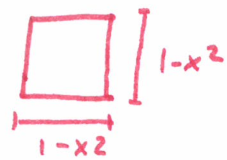
~~Study Session~~ — Spring 2018
Thursday Discussion

1a] $x+1 = x^2+x \implies 0 = x^2-1 \implies (x-1)(x+1)=0$

$\implies x = 1, -1$

$f(0) = 1 > 0 = g(0)$





$$A(x) = (1-x^2)^2 = x^4 - 2x^2 + 1$$

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} x^4 - 2x^2 + 1\,dx = \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^{1}$$

$$= \left(\frac{1}{5} - \frac{2}{3} + 1\right) - \left(-\frac{1}{5} + \frac{2}{3} - 1\right) = \left(\frac{8}{15}\right) - \left(\frac{-8}{15}\right) = \frac{16}{15}$$

1b] $$L = \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$x'(t) = 6t$

$y'(t) = 6t^2$

$[x'(t)]^2 = 36t^2$

$[y'(t)]^2 = 36t^4$

$[x'(t)]^2 + [y'(t)]^2 = 36t^4 + 36t^2$

$= 36t^2(t^2+1)$

$$= \int_0^2 \sqrt{36t^2(t^2+1)}\,dt$$

$$= \int_0^2 6t\sqrt{t^2+1}\,dt$$

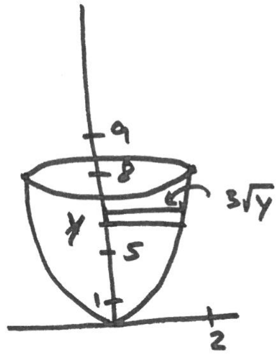
$x = t^2+1$

$dx = 2t\,dt$

$$= \int_1^5 \cancel{36}\, 3\sqrt{x}\,dx$$

$$= \left[2(x)^{3/2}\right]_1^5 = 2(5)^{3/2} - 2$$

2a]



$$A(y) = \pi r^2$$
$$= \pi (\sqrt[3]{y})^2 = \pi y^{3/2}$$

$$W = \int_1^5 62.5\,(l - y)\,A(y)\,dy$$

$$= \int_1^5 62.5\,(9 - y)\,\pi y^{3/2}\,dy$$

2b]

$$\lim_{x \to 0} \frac{\overbrace{2x^3}^{\to 0}}{\underbrace{x - \sin(x)}_{\to 0}} \stackrel{LH}{=} \lim_{x \to 0} \frac{\overbrace{6x^2}^{\to 0}}{\underbrace{1 - \cos(x)}_{\to 0}} \stackrel{LH}{=} \lim_{x \to 0} \frac{\overbrace{12x}^{\to 0}}{\underbrace{\sin(x)}_{\to 0}}$$

$$\stackrel{LH}{=} \lim_{x \to 0} \frac{12}{\underbrace{\cos(x)}_{\to 1}} = \boxed{12}$$

2c] Indeterminate forms:

$\frac{0}{0}$, 0^0, 1^∞, $\frac{\infty}{\infty}$, $\infty - \infty$

3a] $$f'(x) = \frac{(9+x^2) - \cancel{x}\, x(2x)}{(9+x^2)^2} = \frac{9-x^2}{(9+x^2)^2}$$

$$= \frac{(3-x)(3+x)}{(9+x^2)^2}$$

$-$ $+$ $-$

-3 3

$f'(x) > 0$ for $x \in [-3, 3)$ $I = [-3, 3)$

Note that $f(1) = \frac{1}{9+1} = \frac{1}{10}$ So,

$$(f^{-1})'(1/10) = \frac{1}{f'(1)} = \frac{1}{\frac{9-1}{(9+1)^2}} = \frac{100}{8} = \frac{25}{2}$$

3b] $$g'(x) = \frac{d}{dx}\left[\log_4(x^2+3) + 4^{-x}\right]$$

$$= \frac{d}{dx}\left[\frac{1}{\ln(4)} \ln(x^2+3) + e^{-x\ln(4)}\right]$$

$$= \frac{1}{\ln(4)} \frac{1}{x^2+3} \frac{d}{dx}\left[x^2+3\right] + e^{-x\ln(4)} \frac{d}{dx}\left[-x\ln(4)\right]$$

$$= \frac{1}{\ln(4)} \frac{2x}{x^2+3} + 4^{-x}(-\ln(4))$$

4a) $\displaystyle\int_4^6 \frac{1}{2x^2-16x+40}\,dx = \int_4^6 \frac{1}{2(x^2-8x+20)}\,dx$

$\displaystyle = \int_4^6 \frac{1}{2(x^2-8x+16+4)}\,dx = \frac{1}{2}\int_4^6 \frac{1}{(x-4)^2+4}\,dx$ $u=x-4$, $du=dx$

$\displaystyle = \frac{1}{2}\int_0^2 \frac{1}{u^2+4}\,du = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_0^2$

$\displaystyle = \frac{1}{4}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] = \frac{1}{4}\left[\frac{\pi}{4}-0\right] = \frac{\pi}{16}$

4b) $\displaystyle\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$ $A(x+1)+B = x+4$

$x:\ A = 1$

$1:\ A+B = 4 \implies B=3$

$\displaystyle\int \frac{x+4}{(x+1)^2}\,dx = \int \frac{1}{x+1} + \frac{3}{(x+1)^2}\,dx$ $u=x+1$, $du=dx$

$\displaystyle = \int \frac{1}{u} + \frac{3}{u^2}\,du = \ln|u| - \frac{3}{u} + C = \ln|x+1| - \frac{3}{(x+1)^2} + C$

5a] $\int_1^e x^2 \ln(x)\,dx$

$u = \ln(x)$ $\quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{1}{3}x^3$

$$= \left[\ln(x) \cdot \frac{1}{3}x^3\right]_1^e - \int_1^e \frac{1}{3}x^3 \frac{1}{x}\,dx$$

$$= \left[\frac{1}{3}e^3\right] - \int_1^e \frac{1}{3}x^2\,dx = \frac{1}{3}e^3 - \left[\frac{1}{9}x^3\right]_1^e$$

$$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^3 - \frac{1}{9}\right] = \frac{2}{9}e^3 + \frac{1}{9}$$

5b] $\int \sin^3(t)\cos^5(t)\,dt = \int \sin^2(t)\cos^5(t)\sin(t)\,dt$

$$= \int (1-\cos^2(t))\cos^5(t)\sin(t)\,dt$$

$u = \cos(t)$

$du = -\sin(t)\,dt$

$$= -\int (1-u^2)u^5\,du = -\int u^5 - u^7\,du = \int u^7 - u^5\,du$$

$$= \frac{1}{8}u^8 - \frac{1}{6}u^6 + C = \frac{1}{8}\cos^8(t) - \frac{1}{6}\cos^6(t) + C$$

6a] $\int_1^3 \frac{1}{(x-1)^{2/3}}\,dx \quad = \lim_{a\to 1^+} \int_a^3 \frac{1}{(x-1)^{2/3}}\,dx$

Bounded ness error at $x = 1$.

$\int \frac{1}{(x-1)^{2/3}}\,dx \qquad u = x-1, \quad du = dx$

$= \int \frac{1}{u^{2/3}}\,du = \int u^{-2/3}\,du$

$= 3u^{1/3} + c = 3(x-1)^{1/3} + c$

$= \lim_{a\to 1^+} \left[3(x-1)^{1/3}\right]_a^3$

$= \lim_{a\to 1^+} 3(2)^{1/3} - 3(a-1)^{1/3}$

$= 3(2)^{1/3}$

This integral <u>converges</u>.

6b] $\frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

$$\int_0^2 3x^4\,dx \approx \frac{1}{6}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{1}{6}\left[3(0)^4 + 4\left(3\left(\tfrac{1}{2}\right)^4\right) + 2\left(3(1)^4\right) + 4\left(3\left(\tfrac{3}{2}\right)^4\right) + 3(2)^4\right]$$

7a] $a_n = \frac{1}{n^2}$ $\sqrt{a_n} = \frac{1}{n}$

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series as $p = 2 > 1$

$\sum_{n=1}^{\infty} \sqrt{a_n} = \sum_{n=1}^{\infty} \frac{1}{n}$ ~~converges~~ diverges by p-series as $p = 1 \leq 1$

7b] $$\sum_{n=1}^{\infty} \frac{2^n + 5^n}{10^{n+1}} = \frac{1}{10} \sum_{n=1}^{\infty} \frac{2^n + 5^n}{10^n} = \frac{1}{10} \sum_{n=1}^{\infty} \frac{2^n}{10^n} + \frac{5^n}{10^n}$$

$$= \frac{1}{10}\left(\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \right)$$

$$= \frac{1}{10}\left(\frac{\frac{1}{5}}{1 - \frac{1}{5}} + \frac{\frac{1}{2}}{1 - \frac{1}{2}} \right) = \frac{1}{10}\left(\frac{\frac{1}{5}}{\frac{4}{5}} + \frac{\frac{1}{2}}{\frac{1}{2}} \right)$$

$$= \frac{1}{10}\left(\frac{1}{4} + 1 \right) = \frac{5}{40}$$

7c] $$\sum_{k=1}^{\infty} \frac{(-1)^k e^{-k}}{k^2}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1} e^{-(k+1)}}{(k+1)^2}}{\frac{(-1)^k e^{-k}}{k^2}} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} e^{-k-1}}{(k+1)^2} \frac{k^2}{(-1)^k e^{-k}} \right|$$

$$\cancel{\left(= \lim_{k\to\infty} \left| (-1) e^{-1} \right| \right)} = \lim_{k\to\infty} \left| \frac{k^2}{(k+1)^2} (-1) e^{-1} \right| = \frac{1}{e} \lim_{k\to\infty} \frac{k^2}{k^2+2k+1} \frac{\frac{1}{k^2}}{\frac{1}{k^2}}$$

$$= \frac{1}{e} \lim_{k\to\infty} \frac{1}{1 + \underbrace{\frac{2}{k} + \frac{1}{k^2}}_{\to 0}} = \frac{1}{e} < 1$$

By generalized ratio test, the series converges absolutely.

8a) $f(x) = \sum_{n=1}^{\infty} \frac{\pi}{n\,3^n}\, x^{2n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\pi\, x^{2(n+1)}}{(n+1)\,3^{n+1}}}{\frac{\pi\, x^{2n}}{n\,3^n}} \right| = \lim_{n\to\infty} \left| \frac{\pi\, x^{2n+2}}{(n+1)\,3^{n+1}} \quad \frac{n\,3^n}{\pi\, x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\pi x^2}{3} \quad \frac{n}{n+1} \right| = \frac{\pi x^2}{3} \lim_{n\to\infty} \frac{n}{n+1}\, \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{\pi x^2}{3} \lim_{n\to\infty} \frac{1}{1+\underbrace{\frac{1}{n}}_{\to 0}}$$

$$= \frac{\pi x^2}{3} < 1 \implies x^2 < \frac{3}{\pi} \implies |x| < \frac{\sqrt{3}}{\sqrt{\pi}}$$

By generalized ratio test, the radius of convergence is

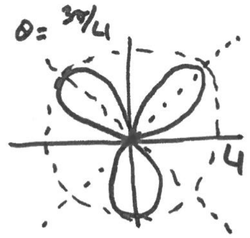
$$R = \frac{\sqrt{3}}{\sqrt{\pi}}$$

8b) $$f'(x) = \frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{\pi}{n\,3^n}\, x^{2n} \right] = \sum_{n=1}^{\infty} \frac{d}{dx}\left[\frac{\pi}{n\,3^n}\, x^{2n} \right]$$

$$= \sum_{n=1}^{\infty} \frac{\pi}{\cancel{n}\,3^n}\, 2\cancel{n}\, x^{2n-1} = \sum_{n=1}^{\infty} \frac{2\pi}{3^n}\, x^{n-1}$$

By a Theorem learned in class, $R_1 = R = \frac{\sqrt{3}}{\sqrt{\pi}}$

9a)



Symmetry w.r.t y-axis

9b)

$$A = \int_0^{\pi/3} \frac{1}{2}\left[4\sin(3\theta)\right]^2 d\theta = \int_0^{\pi/3} 8\sin^2(3\theta)\, d\theta$$

$x = 3\theta$
$dx = 3d\theta$
$d\theta = \frac{1}{3}dx$

$$= \frac{8}{3}\int_0^{\pi} \sin^2(x)\, dx = \frac{8}{3}\int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2x)\, dx$$

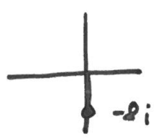
$u = 2x$
$du = 2dx$
$dx = \frac{1}{2}du$

$$= \frac{4}{3}\int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(u)du = \frac{2}{3}\int_0^{2\pi} 1 - \cos(u)\, du$$

$$= \frac{2}{3}\left[u - \sin(u)\right]_0^{2\pi} = \frac{2}{3}\left[2\pi - 0\right] = \frac{4\pi}{3}$$

9c)

$Z = -8i$



$r = 8$ $\quad \theta = 3\pi/2$

$= 8e^{i\frac{3\pi}{2}}$

3rd roots: $8^{1/3}\, e^{i\left(\frac{\pi}{2} + \frac{2\pi k}{3}\right)}$

$$= 2e^{i\left(\frac{3\pi}{6} + \frac{4\pi k}{6}\right)}$$
$$= 2e^{i\left(\frac{(4k+3)\pi}{6}\right)}$$

$Z_0 = 2e^{i\frac{\pi}{2}}$

$Z_1 = 2e^{i\frac{7\pi}{6}}$

$Z_2 = 2e^{i\frac{11\pi}{6}}$