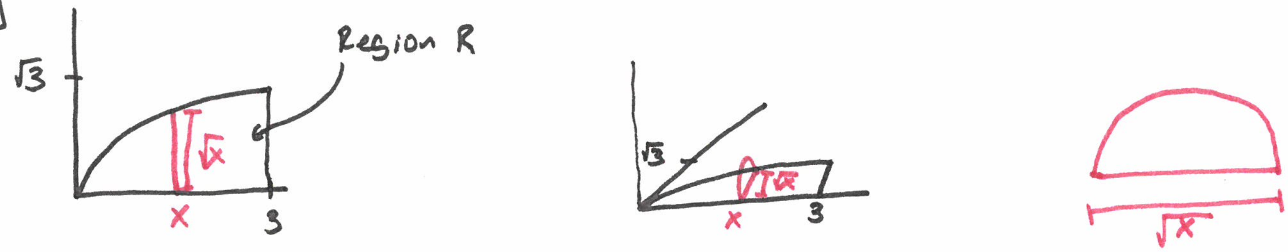
~~Study Session~~ - Spring 2017
Thesday Discussion

1a]



$$A(x) = \frac{1}{2}\pi\left(\frac{\sqrt{x}}{2}\right)^2 = \frac{1}{2}\pi\frac{x}{4} = \frac{1}{8}\pi x$$

$$V = \int_0^3 A(x)\,dx = \int_0^3 \frac{1}{8}\pi x\,dx = \left[\frac{1}{8}\pi\frac{x^2}{2}\right]_0^3$$

$$= \frac{1}{8}\pi\frac{9}{2} - \frac{1}{8}\pi\frac{0}{2} = \frac{9}{16}\pi$$

1b]

$$L = \int_1^2 \sqrt{1+[g'(x)]^2}\,dx$$

$$g(x) = \frac{4}{3}x^{3/2}$$

$$g'(x) = 2x^{1/2}$$

$$[g'(x)]^2 = 4x$$

$$= \int_1^2 \sqrt{1+4x}\,dx$$

$u = 1+4x$

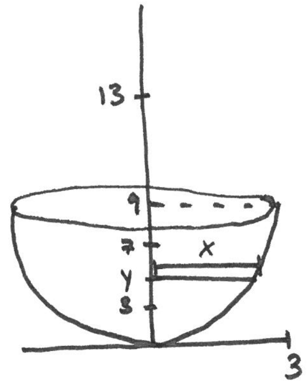
$du = 4dx$

$dx = \frac{1}{4}du$

$$= \frac{1}{4}\int_5^9 \sqrt{u}\,du$$

$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_5^9 = \frac{1}{6}9^{3/2} - \frac{1}{6}5^{3/2} = \frac{9}{2} - \frac{25\sqrt{5}}{6}$$

2a]



$y = x^2$

$x = \sqrt{y} = r$

$A(y) = \pi r^2$

$= \pi(\sqrt{y})^2$

$= \pi y$

$$W = \int_3^7 62.5\ (l - y)\ A(y)\ dy = \int_3^7 62.5\ (13 - y)\ \pi y\ dy$$

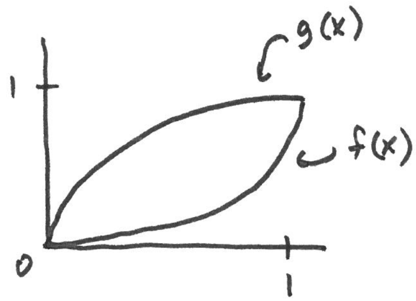
2b] $f(x) = x^2$ $g(x) = \sqrt{x}$

$x^2 = \sqrt{x} \implies x^2 - \sqrt{x} = 0$

$\implies \sqrt{x}\,(x^{3/2} - 1) = 0$

$x = 0, 1$

$f(\frac{1}{4}) = \frac{1}{16} < \frac{1}{2} = g(\frac{1}{4})$



$$M_x = \int_0^1 \frac{1}{2}\left[(g(x))^2 - (f(x))^2\right] dx$$

$$= \int_0^1 \frac{1}{2}\left[x - x^4\right] dx$$

$$M_y = \int_0^1 x\,[g(x) - f(x)]\,dx = \int_0^1 x^{3/2} - x^3\,dx$$

$$A = \int_0^1 g(x) - f(x)\,dx = \int_0^1 \sqrt{x} - x\,dx$$

$\bar{x} = \frac{M_y}{A}$ $\bar{y} = \frac{M_x}{A}$

3a] $f(x) = x^3 - \cos(x) + 5x$ $\quad f'(x) = 3x^2 + \sin(x) + 5$

$3x^2 > 0$, $\sin(x) \geq -1$ so $f'(x) = 3x^2 + \sin(x) + 5 \geq 4$
for all x, thus, f is always increasing

$I = [-\infty, \infty)$. Note that $f(0) = 0^3 - \cos(0) + 5(0) = -1$
$f'(0) = 3(0)^2 + \sin(0) + 5 = 5$

so $(f^{-1})'(-1) = \frac{1}{f'(0)} = \frac{1}{5}$

3b] $g(x) = x^{\cos(x)}$ When $x > 0$, there are no problems
When $x = 0$, $\cos(x) = 1$, so $x^{\cos(x)} = 0^1 = 0$
When $x < 0$, there may be problems taking roots of negative numbers.

So, I will assume that the domain of g is $[0, \infty)$.

$$g'(x) = \frac{d}{dx}\left[x^{\cos(x)}\right] = \frac{d}{dx}\left[e^{\cos(x)\ln(x)}\right] = e^{\cos(x)\ln(x)} \frac{d}{dx}\left[\cos(x)\ln(x)\right]$$

$$= x^{\cos(x)}\left[\frac{d}{dx}[\cos(x)]\ln(x) + \cos(x)\frac{d}{dx}[\ln(x)]\right]$$

$$= x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{1}{x}\cos(x)\right)$$

4a] $$\lim_{x\to\infty} \frac{\underbrace{\ln(\overbrace{x+e^{2x}}^{\to\infty})}_{\to\infty}}{\underbrace{x}_{\to\infty}} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x+e^{2x}}(1+2e^{2x})}{1}$$

$$= \lim_{x\to\infty} \frac{\overbrace{1+2e^{2x}}^{\infty}}{\underbrace{x+e^{2x}}_{\to\infty}} \overset{LH}{=} \lim_{x\to\infty} \frac{4e^{2x}}{1+2e^{2x}} \cdot \frac{\frac{1}{e^{2x}}}{\frac{1}{e^{2x}}}$$

$$= \lim_{x\to\infty} \frac{4}{\underbrace{\frac{1}{e^{2x}}}_{\to 0} + 2} = \frac{4}{2} = 2$$

4b] $$\int \tan^3(2t)\sec^3(2t)\,dt$$

$x = 2t$
$dx = 2dt$
$dt = \frac{1}{2}dx$

$$= \frac{1}{2}\int \tan^3(x)\sec^3(x)\,dx$$

$$= \frac{1}{2}\int \tan^2(x)\sec^2(x)\big(\sec(x)\tan(x)\big)dx$$

$$= \frac{1}{2}\int (\sec^2(x)-1)\sec^2(x)(\sec(x)\tan(x))\,dx$$

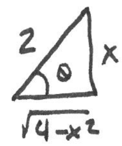
$u = \sec(x)$
$du = \sec(x)\tan(x)\,dx$

$$= \frac{1}{2}\int (u^2-1)(u^2)\,du = \frac{1}{2}\int u^4 - u^2\,du = \frac{1}{2}\left[\frac{u^5}{5} - \frac{u^3}{3}\right] + C$$

$$= \frac{u^5}{10} - \frac{u^3}{6} + C = \frac{1}{10}\sec^5(x) + \frac{1}{6}\sec^3(x) + C$$

$$= \frac{1}{10}\sec^5(2t) + \frac{1}{6}\sec^3(2t) + C$$

5a] $\int \sqrt{4-x^2}\, dx$



$= \int (2\cos\theta)(2\cos\theta)\, d\theta$

$\frac{x}{2} = \sin\theta$ $\qquad \frac{\sqrt{4-x^2}}{2} = \cos\theta$

$= 4\int \cos^2\theta\, d\theta$

$x = 2\sin\theta$ $\qquad \sqrt{4-x^2} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$= 4\int \frac{1}{2}(1+\cos(2\theta))\, d\theta$

$= 2\int 1+\cos(2\theta)\, d\theta = 2\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C = 2\theta + \sin(2\theta) + C$

$= 2\theta + 2\sin\theta\cos\theta + C$

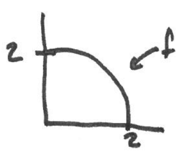
$= 2\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C$

$\int_0^2 \sqrt{4-x^2}\, dx = \left[2\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2}\right]_0^2 = 2\arcsin(1) + \frac{1}{2}\,2\sqrt{4-4}$

$- 2\arcsin(0) + \frac{1}{2}\,0\sqrt{4-0}$

$= 2\arcsin(1) = \pi$

OR:



$\int_0^2 \sqrt{4-x^2}\, dx = \frac{1}{4}\left(\pi (2)^2\right) = \pi$

5b] $\frac{2x+5}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$

$A(x+2) + B = 2x+5$

$x: \quad A = 2$

$1: \quad 2A + B = 5 \Rightarrow 4 + B = 5 \Rightarrow B = 1$

$\int \frac{2x+5}{(x+2)^2}\, dx = \int \frac{2}{x+2}\, dx + \int \frac{1}{(x+2)^2}\, dx$

$u = x+2$

$du = dx$

$= \int \frac{2}{u}\, du + \int \frac{1}{u^2}\, du = 2\ln|u| - \frac{1}{u^2} + C$

$= 2\ln|x+2| - \frac{1}{(x+2)^2} + C$

6a] $\int_{-2}^{-1} \frac{1}{y^2+4y+7}\,dy = \int_{-2}^{-1} \frac{1}{(y^2+4y+4)+3}\,dy = \int_{-2}^{-1} \frac{1}{(y+2)^2+3}\,dy$

$x = y+2$

$dx = dy$

$= \int_{0}^{1} \frac{1}{x^2+3}\,dx = \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$

$= \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan(0)\right] = \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{\sqrt{3}}{3}\right) - 0\right]$

$= \cancel{\left(\frac{1}{\sqrt{3}}\,\frac{\pi}{3} = \frac{\pi}{3\sqrt{3}} = \frac{\pi\sqrt{3}}{9}\right)} = \frac{1}{\sqrt{3}}\,\frac{\pi}{6} = \frac{\pi}{6\sqrt{3}} = \frac{\pi\sqrt{3}}{18}$

6b] When $x > \frac{1}{2}$

$x^2 < x^2+1$

$x^6 < (x^2+1)^3$

$\frac{1}{x^2} = \frac{x^4}{x^6} > \frac{x^4}{(x^2+1)^3}$

$\int_{1/2}^{\infty} \frac{1}{x^2}\,dx = \underbrace{\int_{1/2}^{1} \frac{1}{x^2}\,dx}_{\text{proper integral}} + \underbrace{\int_{1}^{\infty} \frac{1}{x^2}\,dx}_{\text{converges by p-test}}$

$\int_{1/2}^{\infty} \frac{1}{x^2}\,dx$ converges so, by comparison, so does $\int_{1/2}^{\infty} \frac{x^4}{(1+x^2)^3}$

7a] $\lim_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+4}-\sqrt{n}\right)\right) \quad \dfrac{\sqrt{n+4}+\sqrt{n}}{\sqrt{n+4}+\sqrt{n}}$

$$= \lim_{n\to\infty}\left(\frac{\sqrt{n}\,(n+4-n)}{\sqrt{n+4}+\sqrt{n}}\right) = \lim_{n\to\infty}\frac{4\sqrt{n}}{\sqrt{n+4}+\sqrt{n}} \quad \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{4}{\sqrt{1+\underbrace{\tfrac{4}{n}}_{\to 0}}+1} = \frac{4}{1+1} = 2$$

7b] $\sum_{n=1}^{\infty} (-1)^{n+1}\dfrac{2}{n^2}$

Note that $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\dfrac{2}{n^2}\right| = \sum_{n=1}^{\infty}\dfrac{2}{n^2} = 2\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ converges

by p-series as $p = 2 > 1$. So, the series converges absolutely.

$$E_j \leq \frac{2}{(j+1)^2} \overset{\text{want}}{\leq} \frac{1}{8} \qquad \text{so,} \quad \frac{2}{(j+1)^2} \leq \frac{1}{8}$$

$$16 \leq (j+1)^2$$
$$4 \leq j+1$$
$$3 \leq j$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n^2} \overset{\text{within } \frac{1}{8}}{\approx} \sum_{n=1}^{3}(-1)^{n+1}\frac{2}{n^2} = \frac{2}{1^2} - \frac{2}{2^2} + \frac{2}{3^2}$$

$$= 2 - \frac{1}{2} + \frac{2}{9}$$

$$= \frac{36}{18} - \frac{9}{18} + \frac{4}{18} = \frac{31}{18}$$

8a) $g(x) = \sum_{n=1}^{\infty} \frac{n+1}{n(3^{n+1})} x^{2n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{n+2}{(n+1)(3^{n+2})} x^{2(n+1)}}{\frac{n+1}{n(3^{n+1})} x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+2 \quad x^{2n+2}}{(n+1)(3^{n+2})} \quad \frac{n(3^{n+1})}{(n+1)x^{2n}} \right| = \lim_{n\to\infty} \frac{n^2+2n}{(n^2+2n+1)\,3} x^2$$

$$= \frac{x^2}{3} \lim_{n\to\infty} \frac{n^2+2n}{n^2+2n+1} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{x^2}{3} \lim_{n\to\infty} \frac{1 + \overbrace{\frac{2}{n}}^{\to 0}}{1 + \underbrace{\frac{2}{n} + \frac{1}{n^2}}_{0}} = \frac{x^2}{3}$$

To converge for the ratio test,

$$\frac{x^2}{3} < 1 \Rightarrow x^2 < 3 \Rightarrow |x| < \sqrt{3} \qquad \text{so } R = \sqrt{3}$$

$$g'(x) = \frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{n+1}{n(3^{n+1})} x^{2n} \right] = \sum_{n=1}^{\infty} \frac{d}{dx}\left[\frac{n+1}{n(3^{n+1})} x^{2n} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2n(n+1)}{n(3^{n+1})} x^{2n-1} = \sum_{n=1}^{\infty} \frac{2(n+1)}{3^{n+1}} x^{2n}$$

8b) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \frac{1}{1+2x^3} = \sum_{n=0}^{\infty} (2x^3)^n = \sum_{n=0}^{\infty} 2^n x^{3n}$

$$\frac{x}{1+2x^3} = x \sum_{n=0}^{\infty} 2^n x^{3n} = \sum_{n=0}^{\infty} 2^n x^{3n+1}$$

$$\int_0^c f(t)\,dt = \int_0^c \sum_{n=0}^{\infty} 2^n t^{3n+1}\,dt = \sum_{n=0}^{\infty} 2^n \left(\int_0^c t^{3n+1}\,dt \right)$$

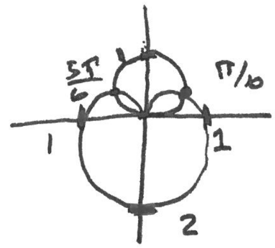
$$= \sum_{n=0}^{\infty} 2^n \left[\frac{1}{3n+2} t^{3n+2} \right]_0^c = \sum_{n=0}^{\infty} \frac{2^n}{3n+2} c^{3n+2}$$

9a) $\sin\theta = 1-\sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

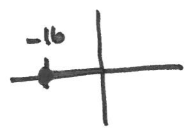
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$



$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(\sin^2\theta - (1-\sin\theta)^2\right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(1 - 2\sin\theta\right) d\theta$$

9b) $z = -16$



$= 16e^{i\pi}$

4th roots of z : $16^{1/4} e^{i\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)} = 2e^{i\left(\frac{\pi}{4} + 2\pi k\right)}$

$= 2e^{i\left(\frac{(2k+1)\pi}{4}\right)}$

$k=0$: $z_0 = 2e^{i\frac{\pi}{4}}$ $\quad$ $k=1$: $z_1 = 2e^{i\frac{3\pi}{4}}$

$k=2$: $z_0 = 2e^{i\frac{5\pi}{4}}$ $\quad$ $k=3$: $z_3 = 2e^{i\frac{7\pi}{4}}$